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Type II/F-theory Superpotentials with Several Deformations and $\mathcal{N}=1$ Mirror Symmetry

Murad Alim[1], **Michael Hecht**[2], **Hans Jockers**[3,4], **Peter Mayr**[2], **Adrian Mertens**[2] **and Masoud Soroush**[2]

[1] *Jefferson Physical Laboratory, Harvard University, Cambridge, MA, 02138, USA*
[2] *Arnold Sommerfeld Center for Theoretical Physics, LMU, Theresienstr. 37, D-80333 Munich, Germany*
[3] *Department of Physics, Stanford University, Stanford, CA 94305-4060, USA*
[4] *Kavli Institute for Theoretical Physics, Santa Barbara, CA 93106, USA*

Abstract: We present a detailed study of D-brane superpotentials depending on several open and closed-string deformations. The relative cohomology group associated with the brane defines a generalized hypergeometric GKZ system which determines the off-shell superpotential and its analytic properties under deformation. Explicit expressions for the $\mathcal{N}=1$ superpotential for families of type II/F-theory compactifications are obtained for a list of multi-parameter examples. Using the Hodge theoretic approach to open-string mirror symmetry, we obtain new predictions for integral disc invariants in the A model instanton expansion. We study the behavior of the brane vacua under extremal transitions between different Calabi-Yau spaces and observe that the web of Calabi-Yau vacua remains connected for a particular class of branes.

Contents

1. Introduction

String backgrounds with mirror symmetry offer some of the rare occasions, where quantitative non-perturbative data on semi-realistic string theory compactifications can be obtained explicitly. An important example are $\mathcal{N}=1$ supersymmetric compactifications of type II strings on Calabi–Yau threefolds with branes, sometimes related to an F-theory compactification on a Calabi–Yau fourfold. The application of open-string mirror symmetry to this case has been pioneered in refs. [1, 2, 3] and in particular in ref. [4], where the authors defined a large class of mirror pairs of brane geometries and obtained the first prediction for Ooguri–Vafa invariants [5] in non-compact geometries from a B-model computation.

Motivated and guided by the results of ref. [4], a Hodge theoretic approach to the computation of D-brane superpotentials and open-string mirror symmetry was put forward in refs. [6, 7]. It was argued, that the periods on the relative cohomology group associated with a B-type brane determine the superpotential as the solution of a GKZ generalized hypergeometric system and that the Hodge filtration defines the mirror map and the potentials for the A model instantons, much as in the case of closed-string mirror symmetry treated in refs. [8, 9, 10, 11]. The Hodge theoretic framework applies also to compact geometries. In refs. [12, 13, 14] the first results on compact manifolds were obtained by computing the dependence of the superpotential on closed-string deformations for rigid branes from so-called normal functions. The case with open-string deformations has been solved in refs. [15, 16, 17] in the relative cohomology framework. Since the open-string degrees of freedom are frozen at a critical value in the first formalism, while the superpotential still depends on open-string deformations away from a critical value in the second formalism, we refer to the two cases as on- and off-shell approaches, respectively.

The off-shell superpotential for the B-type brane geometry on the threefold is often related to the GVW flux superpotential [18] for an M/F-theory compactifications on a dual Calabi-Yau fourfold X_4 by an open-closed duality [19, 16, 20]. The GVW superpotential on the fourfold X_4 can be computed from the integral fourfold periods by standard methods [21, 22, 23, 24] and it agrees with the brane superpotential on the threefold at lowest order in g_s [16, 17, 20, 25, 26]. The full F-theory superpotential computes $\mathcal{O}(g_s), \mathcal{O}(e^{-1/g_s})$ corrections to the superpotential of the local brane geometry [27] and captures the superpotential of dual type II and heterotic compactifications on generalized Calabi-Yau manifolds [28, 29, 27].

The continuous brane deformations test off-shell directions of the superpotential in the open-string direction away from the critical point. Depending on the behavior of the superpotential near the critical locus this leads to two qualitatively different types of instanton expansions in the mirror A-model. Generically, the open-string deformations are obstructed classically and should be integrated out. Freezing the open-string parameters one obtains an instanton expansion of the critical superpotential in the closed-string moduli only, which leads to the modified disc invariants defined in refs. [12, 14]. The other case is a critical locus with almost flat directions also in the open-string direction, where the A model potential has an instanton expansion in closed- *and* open-string deformation parameters. This led to the first B model predictions for genuine Ooguri–Vafa invariants in compact

brane geometries in refs. [16, 30, 17], generalizing the familiar large volume expansion of the closed-string mirror symmetry to the open-string sector.

For compact geometries, the predictions on off-shell superpotentials and invariants obtained from the generalized GKZ systems for relative cohomology pass some non-trivial consistency checks, but await for a verification by independent methods.[1] In this note we further test the Hodge theoretic approach in the more general situation of compact brane geometries with several deformations, near the critical points of the first, generic type. Explicit expressions for the $\mathcal{N}=1$ superpotential for brane compactifications on Calabi-Yau threefolds and related F-theory compactification on Calabi-Yau fourfolds are obtained for these examples. Particular emphasis is given to the relation of the off-shell approach of refs. [16, 30, 17] and the on-shell computations of refs. [12, 13].[2] Specifically the multi-parameter examples studied below lead, at the critical locus, to a class of Picard-Fuchs equations with complicated inhomogeneous pieces given by hypergeometric series with special properties. Applying the mirror map to these examples we obtain new predictions for integral disc invariants on the A model side.

The multi-parameter models allow us to study the fate of the domain walls under extremal transitions between closed-string compactifications on different manifolds, which are believed to connect the web of $\mathcal{N}=2$ vacua represented by different Calabi-Yau manifolds [36, 37]. It is an interesting question to what extent the $\mathcal{N}=2$ web remains connected after adding D-branes. This was already studied in ref. [35] in one example. We find that for extremal transitions through points of enhanced non-Abelian gauge symmetries, the two vacuum branches stay connected for a particular set of domain walls and there is an interesting physical and group theoretic structure. If G denotes the non-perturbative gauge group, the domain walls fall into representations of the Weyl group, with the disc invariants of the domain walls mapping to each other under the group action. At the locus of gauge symmetry enhancement, the domain wall tensions in non-trivial representations degenerate, which implies the existence of tensionless domain walls at this point.

The organization of this note is as follows. In sect. 2 we outline the Hodge theoretic approach to the computation of type II and F-theory superpotentials and describe how the off-shell approach based on families of relative cohomology groups reduces at the critical points to the formalism of normal functions studied in refs. [12, 13]. The crucial link is provided by a subset of the period integrals defined by the relative cohomology group. These determine the critical set as the vanishing locus of a certain period vector and induce an inhomogeneous term in the Picard-Fuchs equations upon restriction to the critical point. We describe the generalized GKZ type systems that annihilate the type II/F-theory superpotential for brane geometries in toric hypersurfaces. In sect. 3 we turn to a detailed study of critical points of the massive type for a number of brane geometries with several parameters. We compute the type II/F-theory superpotential and disc invariants for these vacua and study the fate of the domain walls through extremal transitions to other Calabi–Yau manifolds. In sect. 4 we present our conclusions. Finally in app. A we collect some additional material, which supplements the analysis of the main text. Here we give a

[1] See ref. [31] for recent progress from matrix factorizations.

[2] See also refs. [32, 33, 34, 35] for further examples and discussions.

description of the studied threefolds and fourfolds for type II/F-theory compactifications as toric hypersurfaces. We also study local limits of the compact Calabi-Yau manifolds in the examples. For these local geometries, which can be associated to elliptic curves, we extract disc invariants. These invariants are related to a subset of disc invariants of the corresponding compact Calabi-Yau manifolds.

2. Relative cohomology, generalized GKZ systems and superpotentials

2.1 Brane and flux superpotentials in type II and F-theory

In this note we study the $\mathcal{N}=1$ superpotential $\mathcal{W}$ of B-type D-branes wrapped on even-dimensional cycles of a Calabi-Yau threefold X and, by open-string mirror symmetry, the superpotential of the A brane geometry related to it. The B model compactification will also be related to an F-theory compactification on a dual Calabi-Yau fourfold X_4. In the Hodge theoretic approach of refs. [6, 7, 15, 16, 17], the superpotential $\mathcal{W}$ of these theories is derived from the period integrals $\underline{\Pi}(z,\hat{z})$ on the relative cohomology groups defined by the branes, schematically

$$\underline{\Pi}(z,\hat{z}) = \mathcal{W}(C) - \mathcal{W}(C_*) = \mathcal{W}_{brane}(z,\hat{z})\,, \tag{2.1}$$

where $(z,\hat{z})$ are certain local coordinates on the open-closed deformation space $\mathcal{M}$ specified below,[3] C is a 2-cycle in X wrapped by the D-brane and C_* is a reference cycle in the same homology class, $[C_*]=[C]$. The above expression is equal to the tension of a domain wall interpolating between the configurations obtained by wrapping the D-brane either on C or on C_*. The relative periods also capture the 3-form flux superpotential $W_{flux}=\int_X G\wedge\Omega$ of refs. [18, 38], leading to a unified expression of the four-dimensional superpotential in terms of a general linear combination of all relative period integrals [6, 7]:

$$\mathcal{W}_{\mathcal{N}=1}(z,\hat{z}) = \sum \underline{N}_\Sigma \underline{\Pi}_\Sigma(z,\hat{z}) = \mathcal{W}_{flux}(z) + \mathcal{W}_{brane}(z,\hat{z})\,, \tag{2.2}$$

The coefficients $\underline{N}_\Sigma$ are determined by the topological charges of the brane and flux background. Solving the vacuum condition $\frac{d}{d\hat{z}}\mathcal{W}_{\mathcal{N}=1}=0$ in the open-string direction gives the *on-shell* (in the open-string direction) superpotential $W(z)$ as a function of the closed string moduli and the topological data $\underline{N}_\Sigma$.

To write the *off-shell* superpotential $\mathcal{W}(C)$ on a deformation space $\mathcal{M}$, one needs to specify extra data, in particular a concrete parametrization for the off-shell configurations. The off-shell deformation space for a brane on C is generically infinite dimensional, with most of the deformations representing heavy fields in space-time that should be integrated out. To define a suitable finite dimensional space $\mathcal{M}$ with obstruction potential $\mathcal{W}$ one therefore needs to choose an appropriate set of 'light' fields and integrate out infinitely many others, as is familiar in the effective action approach. The result at the critical locus is independent of the parametrization of the off-shell directions, but the off-shell values depend, in a well-defined way, on the parametrization.

[3]The letters z and $\hat{z}$ are reserved for closed- and open-string deformations, respectively.

Generically, there are many consistent choices for the set of light fields, corresponding to local coordinate patches of the off-shell deformation space of different dimension and range of validity. Each choice of parametrization corresponds to a slightly different formulation as a relative cohomology problem. A preferred class of parametrizations favoured equally well by mathematics and physics arises from the following construction motivated by duality to M/F-theory. Embed C into a 4-cycle D and define $\mathcal{M}$ as the *unobstructed* deformation space of a holomorphic family $\mathcal{D}$ of such 4-cycles. Adding a D-brane charge on $C \subset H_2(D)$ induces a superpotential $\mathcal{W}(C)$ on $\mathcal{M}$ [39, 4]. Physicswise this can be viewed as perturbing the true moduli space $\mathcal{M}$ of an F-theory compactification with an unobstructed family of D-branes wrapped on the 4-cycles D by adding a D-brane charge on a 2-cycle C in D [17, 20]. It was already observed in refs. [6, 7, 15, 16], that this class of parametrizations is the one preferred by the topological open-closed string theory, as it leads to flat coordinates on the open-closed deformation space $\mathcal{M}$, which are in agreement with the expectations from the chiral ring in the topological string theory. Moreover the Hodge theoretic definition of the open-string mirror map obtained in this way yields consistent results for the A model disc invariants, in agreement with localization computations in the A model, if available. Mathematically, this class of parametrizations derives directly from the on-shell meaning of the superpotential as an Abel-Jacobi invariant measuring rational equivalence of the cycles C and C_*, as explained in sect. 2.1 below.

The perturbation idea becomes obvious in the framework of the dual M/F-theory compactification on a related fourfold X_4, which geometrizes the branes to flux [19, 16, 20]. In this context, $\mathcal{M}$ maps to the *unobstructed* complex structure moduli space $\mathcal{M}_{CS}(X_4)$ of the fourfold X_4, which is the vacuum space of topological strings in the type IIA compactification on X_4, and open-closed mirror symmetry maps to closed-string mirror symmetry for fourfolds. Adding a 4-form flux G induces the Gukov-Vafa-Witten superpotential [18] on the moduli space $\mathcal{M}_{CS}(X_4)$, and this is the dual description of the off-shell deformation space $\mathcal{M}$ of the brane geometry and the obstruction superpotential $\mathcal{W}(C)$ on it. More precisely, the F-theory superpotential[4] on X_4 computes g_s corrections to the superpotential $\mathcal{W}(C)$ as captured by the relation [17, 27]

$$\mathcal{W}_{GVW}(X_4) = \int_{X_4} G \wedge \Omega^{(4,0)} = \sum_{\Sigma} N_{\Sigma}(G)\ \underline{\Pi}(z, \hat{z}) + \mathcal{O}(g_s) + \mathcal{O}(e^{-1/g_s})\,, \tag{2.3}$$

where the leading term on the right hand side is the result (2.2) for the B-type branes on the threefold with the linear combination of relative periods determined by the flux G on the fourfold. We will only consider the leading term in g_s in this paper, which can be computed from the integral periods of a certain non-compact limit $X_4^\sharp$ of X_4, related to the threefold X by the open-closed duality [19, 16, 20]. The details of the compactification X_4 of $X_4^\sharp$ affect only the higher terms in g_s and can be computed similarly [27]. More details and many examples on the computation of the fourfold superpotential from the geometric period integrals can be found in refs. [22, 23, 24].

[4] See ref. [20] for the discussion from the M-theory perspective.

2.2 Relative periods and domain wall tensions

As alluded to above, a preferred parametrization adapted to topological string states and open-string mirror symmetry is to parametrize the off-shell deformations of the D-brane on a 2-cycle C by the deformations of a holomorphic family of 4-cycles $\mathcal{D}$ that embed $C \in H_2(D)$. The relative periods capturing the superpotential for the brane on C are obtained by restriction to the subspace $H_3(X, C) \subset H_3(X, D)$. Mathematically, this class of parametrizations derives directly from the concept of rational equivalence and the on-shell meaning of the superpotential as an Abel-Jacobi invariant, as will be discussed now.[5]

To this end, consider a Calabi-Yau threefold X_0 together with an ample divisor D_0. We assume that $H^{1,0}(X_0) = H^{2,0}(X_0) = 0$, such that the complex structure deformations of the pair (X_0, D_0) are unobstructed. Then this pair (X_0, D_0) extends to a family of Calabi-Yau threefolds together with a family of ample divisors $\pi : (\mathcal{X}, \mathcal{D}) \to \Delta$ fibered over the disc Δ, which parametrizes a local patch of the combined moduli space $\mathcal{M}$ of the family obtained by deforming the central fiber $\pi^{-1}(0) = (X_0, D_0)$. $\mathcal{M}$ is a fibration $\hat{\mathcal{M}} \to \mathcal{M} \to \mathcal{M}_{CS}$, where the base $\mathcal{M}_{CS}$ corresponds to complex structure deformations z of the family of Calabi-Yau threefolds $\mathcal{X}$, while the fiber $\hat{\mathcal{M}}$, parametrized by the coordinates $\hat{z}$, corresponds to the deformations of the family of divisors $\mathcal{D}$. In string theory, the former arise in the closed-string sector and the latter in the open-string sector.

Since the holomorphic three form $\Omega(z)$ of the Calabi-Yau threefold X_z vanishes on the divisor $D_{(z,\hat{z})}$, the three form $\Omega(z)$, which is an element of $H^3(X_z)$, lifts to an element $\underline{\Omega}(z,\hat{z})$ of the relative cohomology group $H^3(X_z, D_{(z,\hat{z})})$. We define the integral relative periods as[6]

$$\underline{\Pi}(\underline{\Gamma}; z, \hat{z}) = \int_{\underline{\Gamma}_{(z,\hat{z})}} \underline{\Omega}(z, \hat{z}) \ , \tag{2.4}$$

where $\underline{\Gamma}_{(z,\hat{z})}$ is an integral relative cycle in $H_3(X_z, D_{(z,\hat{z})}, \mathbb{Z})$ whose boundary $\partial\underline{\Gamma}_{(z,\hat{z})}$ is trivial as a class in $H_2(X, \mathbb{Z})$. For concreteness we often assume that the boundary is the difference $\partial\underline{\Gamma}_{(z,\hat{z})} = C^+_{(z,\hat{z})} - C^-_{(z,\hat{z})}$ of two 2-cycles in $H_2(D_{z,\hat{z}})$ with $[C^+_{(z,\hat{z})}] = [C^-_{(z,\hat{z})}]$ in $H_2(X_z, \mathbb{Z})$.

Following the fundamental works [4, 39], it was proposed in refs. [6, 7, 15] that the relative period (2.4) defines the off-shell tension $\mathcal{T}(z,\hat{z})$ of a physical D-brane wrapped on the chain $\underline{\Gamma}_{(z,\hat{z})}$, that is $\mathcal{T}(z,\hat{z}) = \underline{\Pi}(\underline{\Gamma}; z, \hat{z})$. This D-brane represents a domain wall interpolating between the two configurations obtained by wrapping a D-brane on $C^+_{(z,\hat{z})}$ or on $C^-_{(z,\hat{z})}$ and its tension measures the difference of the value of the superpotentials for the two D-brane configurations

$$\mathcal{T}(z,\hat{z}) = \mathcal{W}(C^+_{(z,\hat{z})}) - \mathcal{W}(C^-_{(z,\hat{z})}) \ . \tag{2.5}$$

The vacuum condition in the open-string direction is $\frac{d}{d\hat{z}}\mathcal{W}(C^\pm)|_{\hat{z}=\hat{z}_{crit}} = 0$ and it holds if $C^\pm_z := C^\pm_{(z,\hat{z}_{\rm crit})}$ is a holomorphic curve [39]. Imposing this condition on both branes implies $\frac{d}{d\hat{z}}\mathcal{T}(z,\hat{z}) = 0$ as well.

[5] For a related mathematical discussion, see ref. [40].

[6] Objects defined in relative (co-)homology will be distinguished by an underline.

Mathematically speaking, the vacuum configurations hence lie within the so-called Noether-Lefshetz locus, defined as [41]

$$\mathcal{N} \;=\; \left\{ (z,\hat z) \in \Delta \;\middle|\; 0 \,\equiv\, \frac{d\underline{\Pi}(z,\hat z)}{d\hat z} \right\} \; . \tag{2.6}$$

Equivalently the locus $\mathcal{N}$ can be specified by the vanishing condition

$$\mathcal{N} \;=\; \left\{ (z,\hat z) \in \Delta \;\middle|\; 0 \,\equiv\, \vec\pi(z,\hat z;\partial\underline{\Gamma}_{(z,\hat z)}) \right\} \; , \tag{2.7}$$

for the period vector of the divisor $D_{(z,\hat z)}$

$$\vec\pi(z,\hat z;\partial\underline{\Gamma}_{(z,\hat z)}) \;=\; \left(\int_{\partial\underline{\Gamma}_{(z,\hat z)}} \omega^{(2,0)}_{\hat a}(z,\hat z) \right) \; , \quad \hat a = 1,\ldots,\dim H^{2,0}(D_{(z,\hat z)}) \; . \tag{2.8}$$

Here $\omega^{(2,0)}_{\hat a}(z,\hat z)$ is a basis of two forms for $H^{2,0}(D_{(z,\hat z)})$. Hence the critical locus of D-brane vacua is mapped to the subslice of complex structures on the surface $D_{(z,\hat z)}$, where certain linear combinations of period vectors on the surface vanish. At such points in the complex structure the Picard lattice of the surface $D_{(z,\hat z)}$ is enhanced due to the appearance of an additional integral $(1,1)$-form.

At the Noether-Lefschetz locus $(z,\hat z_{\rm crit}) \in \mathcal{N}$ there is an interesting connection between the relative periods and another mathematical quantity studied in refs. [12, 13]. By the result of ref. [41], the relative period $\underline{\Pi}(z,\hat z)$ evaluated at the Noether-Lefschetz locus $(z,\hat z_{\rm crit}) \in \mathcal{N}$ gives (modulo bulk periods) the Abel-Jacobi invariant associated to the normal function of the algebraic curve $\partial\underline{\Gamma}_{(z,\hat z_{\rm crit})}$:

$$\underline{\Pi}(z,\hat z_{\rm crit}) \;=\; \nu_{c_2^{\rm alg}(\partial\underline{\Gamma}_{(z,\hat z_{\rm crit})})}(z) \quad \mathrm{mod} \quad (\text{bulk periods}) \; . \tag{2.9}$$

Specifically, the Abel-Jacobi invariant is defined via the normal function $\nu_{c_2^{\rm alg}(\alpha)}(z)$ as

$$AJ \,:\, CH^2(X_z) \to J^3(X_z) \simeq \frac{F^2H^3(X_z)^*}{H_3(X_z,\mathbb{Z})} \, ; \qquad \alpha \mapsto \nu_{c_2^{\rm alg}(\alpha)}(z) \, , \tag{2.10}$$

where, in the concrete setting, the normal function is defined as the chain integral

$$T(z) \;=\; \int_{\Gamma_z^\pm} \Omega(z) \;=\; \nu_{c_2^{\rm alg}(C_z^+ - C_z^-)}(z) \quad \mathrm{mod} \quad (\text{bulk periods}) \; . \tag{2.11}$$

Here $\partial\Gamma_z^\pm = C_z^+ - C_z^-$, with $C_z^\pm$ the holomorphic curves at fixed $\hat z = \hat z_{\rm crit}$. The essential point is that (only) at the critical locus, the above integral is well-defined in absolute cohomology, because the potentially dangerous boundary terms vanish by holomorphicity of the boundary $\partial\Gamma_z^\pm$ and the Hodge type of Ω. The normal functions (2.11) have been introduced in refs. [12, 13] to study the on-shell values of the superpotentials

$$T(z) = W(C_z^+) - W(C_z^-) \; .$$

By the above argument, these are the restrictions of the relative period integrals (2.4) to the critical locus $\mathcal{N}$.

There is also a partial inverse of this relation, which recovers the relative periods for the family of divisors starting from the normal functions. To this end, recall the meaning of rational equivalence and the Abel-Jacobi invariant. The second algebraic Chern class c_2^{alg} takes values in the second Chow group $CH^2(X_z)$, which consists of equivalence classes of algebraic cycles of co-dimension two modulo rational equivalence [42].[7] Two algebraic cycles α and β of co-dimension two are rationally equivalent, if we can find a subvariety V of co-dimension one, in which α and β are rationally equivalent as co-dimension one cycles. This is the case if α and β are given by two linearly equivalent divisors on V, that is $[\alpha - \beta] = 0 \in CH^1(V)$.[8] Moreover, rational equivalence implies that the Abel-Jacobi invariant vanishes.

Starting from an algebraic cycle α of co-dimension two with $c_2^{\text{top}}(\alpha) = 0$ we can find a three chain Γ^α such that $\alpha = \partial\Gamma^\alpha$, and associate a normal function $\nu_{c_2^{\text{alg}}(\alpha)}$ to it via the integral (2.11). By (2.10), the normal function vanishes for algebraic two cycles $C_z^\pm$ that are rationally equivalent [13]. On the contrary, if C_z^+ and C_z^- are not rationally equivalent, we obtain an element in the relative cohomology of each family $\mathcal{D}(z,\hat{z})$ of divisors that contains the two holomorphic curves $C_z^\pm$ at a 'critical' value $\hat{z} = \hat{z}_{\text{crit}}$. Indeed, since C_z^+ and C_z^- are *not* rationally equivalent, $C_z^+ - C_z^-$ defines by Poincaré duality a non-trivial Element $\omega \in \text{Pic}(D_{(z,\hat{z}_{\text{crit}})}) \simeq H^{1,1}(D_{(z,\hat{z}_{\text{crit}})}) \cap H^2(D_{(z,\hat{z}_{\text{crit}})}, \mathbb{Z})$. Since the algebraic cycle α is topologically trivial on X_z, the associated two form ω is *not* induced from the hypersurface X_z and lifts to a relative three form $\underline{\Theta}_{(z,\hat{z}_{\text{crit}})}$ by the relation

$$H^3(X_z, D_{(z,\hat{z})}) \simeq \text{coker}\left(i^* : H^2(X_z) \to H^2(D_{(z,\hat{z})})\right) \oplus \ker\left(i^* : H^3(X_z) \to H^3(D_{(z,\hat{z})})\right) ,$$

with $i : D_{(z,\hat{z})} \hookrightarrow X_z$. By construction, the three-chain $\Gamma^\alpha \simeq \underline{\Gamma}_{(z,\hat{z}_{\text{crit}})}$ is a representative of the relative homology class in $H_3(X_z, D_{(z,\hat{z}_{\text{crit}})})$ dual to $\underline{\Theta}_{(z,\hat{z}_{\text{crit}})}$. Surjectivity of the boundary map of homology then asserts that the above construction assigns to each normal function a relative period on $H^3(X_z, D_{(z,\hat{z})})$, which measures the superpotential of the off-shell deformation parametrized by the family $\mathcal{D}(z,\hat{z})$.

The relative (co-)homology groups $H_3(X_z, D_{(z,\hat{z}_{\text{crit}})}, \mathbb{Z})$ (and $H^3(X_z, D_{(z,\hat{z}_{\text{crit}})}, \mathbb{Z})$) are topological and do not depend on the open-closed deformation parameters, for a smooth family of the pair $(\mathcal{X}, \mathcal{D})$. As a consequence the relative three cycle $\Gamma_{(z,\hat{z}_{\text{crit}})}$ (and three-form $\underline{\Theta}_{(z,\hat{z}_{\text{crit}})}$) extends over the whole disc $(z,\hat{z}) \in \Delta$. Therefore we can define a relative three cycle $\Gamma_{(z,\hat{z})}$ (and a relative three form $\underline{\Theta}_{(z,\hat{z})}$) for all open parameters $\hat{z}$ and study the relative period integrals $\underline{\Pi}(z,\hat{z})$ using the Mixed Hodge Variation on the family of relative cohomology groups over Δ. The Gauss-Manin derivative on this local system provides a powerful framework to study the relative periods as solutions to a system of Picard-Fuchs equations and leads to a predictive proposal for off-shell mirror symmetry formulated in refs. [6, 7, 15, 16, 17].

Using this connection between normal functions (2.11), that is to say domain walls between critical points $C_z^\pm$, and the off-shell tensions represented by the integral relative

[7] The second algebraic Chern class is a refined invariant of the topological second Chern class [42].

[8] If the subvariety V is not normal the cycles α and β are rationally equivalent, if their Weil divisors D_α and D_β are linearly equivalent in the normalization $\tilde{V}$ of V, namely $\alpha \sim \beta$ if $D_\alpha \sim D_\beta$ with $f : \tilde{V} \to V$ and $\alpha = f_* D_\alpha$ and $\beta = f_* D_\beta$.

periods (2.4) ending on $C^{\pm}_{(z,\hat{z})}$, we may calculate the critical tensions as follows. First determine the possible critical points as the vanishing locus (2.7) of the periods of the surface $D_{(z,\hat{z})}$. The critical domain wall tension is then given by the relative period associated with the vanishing period on the surface, evaluated at the critical point z_{crit}

$$T(z) \,=\, \mathcal{T}(z,\hat{z}_{\text{crit}})\,. \tag{2.12}$$

This determines the critical tension up to a possible addition of a bulk period $\Pi_{\text{Bulk}}(z)$.

The vanishing condition (2.7), classifying the critical points, can be studied very explicitly for off-shell deformations in a single open-string parameter $\hat{z}$, which is sufficient to determine the on-shell tensions. In this case the surface $D_{(z,\hat{z})}$ has geometric genus one and it is isogenic to a K3 surface [43], that is the integral Hodge structures of the surface $D_{(z,\hat{z})}$ can be mapped to the equivalent Hodge structure of its isogenic K3 surface. This has already been used in ref. [17] and will simplify the discussion in some of the examples below.

One particular type of solutions to the vanishing condition arises at the discriminant locus of the isogenic K3 surface, where the period vector, associated with a geometrically vanishing cycle in the K3 surface, develops a zero. However, this type of solution is non-generic in the sense that it is often related to points in the deformation space with a domain wall with zero tension. The generic critical points arise instead from a zero of the period vector, which is a linear combination of volumes of geometric cycles in the K3 surface rather then the volume of an irreducible cycle. The typical example is a point where the volumes of two different cycles coincide, such that the period vector associated with the difference vanishes. At these particular symmetric points there is an 'accidental' global symmetry of the K3 lattice, exchanging the two cycles. More generally the generic critical points should be classified by special symmetric points in the K3 moduli studied in ref. [44].

2.3 Generalized GKZ systems and Picard-Fuchs equations for type II/F-theory superpotentials

As alluded to above, the flat Gauss-Manin connection on the relative cohomology bundle leads to a Picard-Fuchs type of differential operators for the relative periods, which provide an effective method to determine and to study the tensions $\mathcal{T}(z,\hat{z})$ [6, 7, 15, 16, 17]. These differential equations also reflect the duality of B-type branes on the threefold X to an M/F-theory compactification on a fourfold X_4 determined by open-closed duality [19, 16, 20]. Specifically, the set of differential operators for the relative periods on X and for the fourfold periods on X_4 have the superpotential periods in eqs. (2.2) and (2.3) as common solutions, and the superpotential can be equivalently computed on the threefold or on the fourfold.

For concreteness, we assume that the holomorphic curves $C^{\pm}_{z}$ are contained in the intersection of the hypersurface $X:\ P=0$ with two hyperplanes $D_{1,2}$ defined in a certain ambient space. Choose coordinates such that the equation for D_1 does not depend on the

closed-string moduli z, typically of the form[9]

$$D_1: \; x_1^a + \eta \, x_2^b = 0 \,,$$

where x_i are some homogeneous coordinates on the ambient space, a, b some constants that depend on the details and η a fixed constant, which is a phase factor in appropriate coordinates. This hyperplane can be deformed into a family $\mathcal{D}_1: \; x_1^a + \hat{z} \, x_2^b = 0$ by replacing the constant η by a complex parameter $\hat{z}$. The relative 3-form $\underline{\Omega}$ and the relative period integrals on the family of cohomology groups $H^3(X, D_1)$, satisfy a set of Picard-Fuchs equations [6, 7, 15, 26]

$$\mathcal{L}_a(\theta, \hat{\theta}) \, \underline{\Omega} = \underline{d\omega}^{(2,0)} \quad \Rightarrow \quad \mathcal{L}_a(\theta, \hat{\theta}) \; \mathcal{T}(z, \hat{z}) = 0 \,, \qquad a = 1, ..., A \,,$$

where a is some label for the operators. The differential operators can be split into two pieces

$$\mathcal{L}_a(\theta, \hat{\theta}) =: \mathcal{L}_a^{bulk} - \mathcal{L}_a^{bdry} \, \hat{\theta} \,, \tag{2.13}$$

where the bulk part $\mathcal{L}_a^{bulk}(\theta)$ acts only on the closed-string moduli z and the boundary part $\mathcal{L}_a^{bdry}(\theta, \hat{\theta}) \, \hat{\theta}$ contains at least one derivative in the parameters $\hat{z}$. Since the dependence on $\hat{z}$ localizes on D_1, the derivatives $2\pi i \, \hat{\theta} \, \mathcal{T}(z, \hat{z})$ are proportional to the periods (2.8) on the surface D_1

$$2\pi i \, \hat{\theta} \, \mathcal{T}(z, \hat{z}) = \pi(z, \hat{z}) \;. \tag{2.14}$$

Rearranging eq.(2.13) and restricting to the critical point $\hat{z} = \eta$ one obtains an inhomogeneous Picard-Fuchs equation

$$\mathcal{L}_a^{bulk} \, T(z) = f_a(z) \,, \tag{2.15}$$

with $T(z) = \mathcal{T}(z, \eta)$ and

$$2\pi i \, f_a(z) = \; \mathcal{L}_a^{bdry} \, \pi(z, \hat{z}) \Big|_{\hat{z}=\eta} \;. \tag{2.16}$$

In absolute cohomology the inhomogeneous term $f_a(z)$ is due to the fact that the bulk operators $\mathcal{L}_a^{bulk}$ satisfy

$$\mathcal{L}_a^{bulk} \, \Omega = d\beta \quad \Rightarrow \quad \mathcal{L}_a^{bulk} \int_{\Gamma \in H_3(X, \mathbb{Z})} \Omega = 0 \;, \tag{2.17}$$

where d is the differential in the absolute setting. This is sufficient to annihilate the period integrals over cycles, as indicated on the right hand side of the above equation, but leads to boundary terms in the chain integral (2.11). In the absolute setting and based on Dwork-Griffiths reduction the inhomogeneous term $f_a(z)$ has been determined by a residue computation in ref. [13]. Here we see that the functions $f_a(z)$ are different derivatives of the surface period $\pi(z, \hat{z})$, restricted to the critical point. Hence, together with the bulk Picard-Fuchs operators, the surface period determine both the critical locus (2.7) and the critical tension.

[9] Note that the equation for D_1 is a priori defined in the ambient space. However, by restriction to the hypersurface X we also identify D_1 with a divisor on the hypersurface X. For ease of notation we denote both the divisor of the ambient space and of the hypersurface with the same symbol D_1.

In the examples we find that the inhomogeneous terms $f_a(z)$ satisfy a hypergeometric differential equation as well:

$$\mathcal{L}_a^{inh}\, f_a(z) = 0\ . \tag{2.18}$$

The hypergeometric operators $\mathcal{L}_a^{inh}$ descend from the Picard-Fuchs operators $\mathcal{L}^{\mathcal{D}}$ of the surface, which annihilate the surface periods $\mathcal{L}^{\mathcal{D}}\pi(z,\hat{z}) = 0$.[10] Specifically, if $f_a(z)$ is non-zero, the operator $\mathcal{L}_a^{inh}$ can be defined as

$$\mathcal{L}_a^{inh} = \left(\mathcal{L}^{\mathcal{D}} + [\mathcal{L}_a^{bdry}, \mathcal{L}^{\mathcal{D}}]\, \mathcal{L}_a^{bdry-1}\right)_{\hat{z}=\eta}\ , \tag{2.19}$$

where the operators on the right hand side are restricted to the critical point as indicated.

It follows from the above that the inhomogeneous terms $f_a(z)$ can be written as an infinite hypergeometric series in the closed-string moduli. However, on general grounds the $f_a(z)$ need to be well-defined over the open-closed moduli space, which simplifies *on-shell* to a finite cover of the complex structure moduli space $\mathcal{M}_{CS}(X)$ of the threefold [35]. This implies that the hypergeometric series $f_a(z)$ can be written as rational functions in the closed string moduli and the roots of the extra equations defining the curves C.[11]

In the examples we observe that already the leading terms of the surface periods $\pi(z,\hat{z})$ become rational functions at the special symmetric points on the Noether-Lefshetz locus $\mathcal{N}$ in this sense. Hence there appears to be a connection between the enhancement of the Picard-lattice of the surface at these points, rationality of its periods and D-brane vacua. The rationality property is preserved when acting with $\mathcal{L}^{bdry}$ in eq. (2.16) to obtain the inhomogeneous term f_a. In the examples we verify, that the contribution $f_a(C_{\alpha_\ell})$ of a particular boundary curve C_{α_ℓ} to the inhomogeneous term can be written in closed form as follows.

$$f_a(C_{\alpha_\ell}) \;=\; \frac{p_a(\psi,\alpha)}{q_a(\psi,\alpha)}|_{\alpha=\alpha_\ell(\psi)} \;=\; \frac{g_a(\psi,\alpha)}{\prod_i \Delta_l(C)^{\gamma_i^a}}|_{\alpha=\alpha_\ell(\psi)}\,, \tag{2.20}$$

where p_a, q_a are polynomials in the variables (ψ,α). Here $\psi = \psi(z)$ is a short-hand for the fractional power of the closed string moduli z appearing in the defining equation of the hypersurface X and $\{\alpha_\ell\}$ are the roots of the extra equations defining the curves, with the root α_ℓ corresponding to the component C_{α_ℓ}. Moreover, the zeros of the denominator appear only at the zeros of the components $\Delta_i(C)$ of the open-string discriminant, where different roots/curves coincide for special values of the moduli ψ. The exponents γ_i^a are some constants and $g_a(\psi,\alpha)$ some functions without singularities in the interior of the moduli space.

For Calabi-Yau hypersurfaces in toric varieties, the differential operators $\mathcal{L}_a$ can be derived from the GKZ type differential operators associated with the toric action on the ambient space [17, 16, 26]. In particular, the holomorphic $(2,0)$ forms $\omega^{(2,0)}$ on D_1 arise from the Lie derivatives of the holomorphic (3,0) form

$$\omega^{(2,0)} = i_{v_{\hat{\theta}}}\Omega|_{D_1},$$

[10] For simplicity we suppress an index for distinguishing several Picard-Fuchs operators $\mathcal{L}^{\mathcal{D}}$.

[11] We are grateful to Johannes Walcher for explaining to us this property of the inhomogeneous terms and for pointing out the results of ref. [45] on this issue.

where $v_{\hat\theta}$ is the vector field generating the toric $\mathbb{C}^*$ action parametrized by $\hat z$, e.g. $(x_1, x_2) \to (\lambda x_1, \lambda^{-1} x_2)$ in the above example. It is not hard to see that in the above situation, the differential operators for relative cohomology of ref. [17] depending on the parameters $\hat z$, reduce at $\hat z = \eta$ to the type of differential operators derived in prop. 3.3. of ref. [26] in the absolute setting, i.e. without open-string deformations. Specifically, the derivative in the parameter $\hat\theta$ becomes equivalent to the Lie derivatives in the direction of x_1 and x_2 at the critical point $\hat z = \eta$.

In the notation of refs. [16, 17, 26], the GKZ system for the relative periods on X (or equivalently the fourfold periods for F-theory compactification on X_4) are expressed in terms of *extended* charge vectors l^a of the gauged linear sigma model (GLSM) associated with the brane geometry $(X, \mathcal{D})$. In the final form these are given by[12]

$$\mathcal{L}(l) = \prod_{k=1}^{l_0}(\vartheta_0 - k) \prod_{l_i>0} \prod_{k=0}^{l_i-1}(\vartheta_i - k) - (-1)^{l_0} z_a \prod_{k=1}^{-l_0}(\vartheta_0 - k) \prod_{l_i<0} \prod_{k=0}^{-l_i-1}(\vartheta_i - k) \,, \tag{2.21}$$

where l is an arbitrary integral linear combination of the extended charge vectors l^a and $\vartheta_i = a_i \frac{\partial}{\partial a_i}$ are logarithmic derivatives with respect to the parameters a_i in the defining equations for X_z and $D_{(z,\hat z)}$. For details we refer to the examples in sect. 3 and to refs. [16, 17, 20, 26]. From the redundant parameters a_i one may define torus invariant algebraic coordinates z_a on the open-closed deformation space $\mathcal{M}$ by

$$z_a = (-)^{l_0^a} \prod_i a_i^{l_i^a} \ , \tag{2.22}$$

where l^a, $a = 1, ..., \dim \mathcal{M}$ is a fixed choice of basis vectors. These describe the $h^{2,1}(X_z)$ complex structure moduli of X_z and in addition the brane deformations $\hat z$, providing coordinates on the fiber of $\hat{\mathcal{M}} \to \mathcal{M}$. For appropriate choice of basis vectors l^a, solutions to the GKZ system can be written in term of the generating functions in these variables as

$$B_{\{l^a\}}(z_a; \rho_a) = \sum_{n_1,\ldots,n_N \in \mathbb{Z}_0^+} \frac{\Gamma\left(1 - \sum_a l_0^a (n_a + \rho_a)\right)}{\prod_{i>0} \Gamma\left(1 + \sum_a l_i^a (n_a + \rho_a)\right)} \prod_a z_a^{n_a + \rho_a} \,. \tag{2.23}$$

Under certain conditions discussed in refs. [19, 17, 20, 26], the extended GKZ system (2.21) for the relative periods for the brane compactification on the threefold X can be associated also to the periods of the non-compact limit $X_4^\sharp$ of the dual fourfold X_4 for M/F-theory compactification. The solutions to this system then describe at the same time the relative period integrals $\underline{\Pi}$, which give rise to the leading term in eq. (2.3), and the periods of the non-compact 4-fold $X_4^\sharp$. A discussion of the quantum corrections in g_s, computed by the periods of the compact fourfold, can be found in ref. [27].

[12]The same formula describes also the generalized hypergeometric operators of GKZ type for the closed-string compactification [10, 11] and this will be used in the examples to determine the periods of the threefold X and the surface $\mathcal{D}$ below. The distinction between the three different cases arises only from the different generators l^a, which encode the action of the gauge symmetry of the GLSM associated with the surface $\mathcal{D}$, the threefold X, the brane geometry $(X, \mathcal{D})$ and the dual F-theory fourfold X_4, respectively, with the latter two cases having the identical generators in the decoupling limit of ref. [17].

3. Examples

We proceed with the study of type II/F-theory superpotentials for a collection of examples of brane geometries on toric hypersurfaces with several open-closed string deformations. Combining the small Hodge variation associated with the surface periods (2.8) and the GKZ system on the relative cohomology group (2.21) provides an efficient method to compute the integral relative period integrals and the mirror map for a large number of deformations. We obtain new enumerative predictions for the A model expansion, consistent with the expectations, and study the behavior of the branes under extremal transitions between different topological manifolds through points with enhanced non-abelian gauge symmetries.

3.1 Degree 12 hypersurface in $\mathbb{P}_{1,2,2,3,4}$

The charge vectors of the GLSM for the A model manifold are given by [11]

$$\begin{array}{c|c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline l^1 & -6 & -1 & 1 & 1 & 0 & 2 & 3 \\ l^2 & 0 & 1 & 0 & 0 & 1 & 0 & -2 \end{array} \ . \tag{3.1}$$

These vectors describe the relations between the vertices of a reflexive polyhedron described in app. A.1. Written in homogeneous coordinates of $\mathbb{P}_{1,2,2,3,4}$ the hypersurface constraint for the mirror manifold reads

$$\begin{aligned} P &= a_1x_1^{12} + a_2x_2^6 + a_3x_3^6 + a_4x_4^4 + a_5x_5^3 + a_0x_1x_2x_3x_4x_5 + a_6x_1^6x_4^2 && (3.2)\\ &= x_1^{12} + x_2^6 + x_3^6 + x_4^4 + x_5^3 + \psi\, x_1x_2x_3x_4x_5 + \phi\, x_1^6x_4^2\ . && (3.3) \end{aligned}$$

In the second equation the variables x_i have been rescaled to display the dependence on the torus invariant parameters $\psi = z_1^{-1/6}z_2^{-1/4}$ and $\phi = z_2^{-1/2}$, with the z_a given by (2.22). On the mirror manifold, the Greene-Plesser orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with weights[13]

$$\mathbb{Z}_6:\ g_1 = (1,-1,0,0,0), \quad \mathbb{Z}_6:\ g_2 = (1,0,-1,0,0)\,, \tag{3.4}$$

where we denote the generators by λ_k with $\lambda_{1,2}^6 = 1$. The closed-string periods near the large complex structure point can be generated by evaluating the functions $B_{\{l^a\}}(z_a;\rho_a)$ in (2.23) and its derivatives with respect to ρ_i at $\rho_1 = \rho_2 = 0$ [11].

In this geometry we consider the set of curves defined by the equations

$$\begin{aligned} C_{\alpha,\kappa} = \{x_2 = \eta x_3\ ,\ x_4 = \alpha x_1^3\ ,\ x_5 = \kappa\sqrt{\alpha\eta\psi}x_3x_1^2\}\,, \\ \eta^6 = -1\,, \qquad \kappa^2 = -1\,, \qquad \alpha^4 + \phi\alpha^2 + 1 = 0\,. \end{aligned} \tag{3.5}$$

[13] The other factors of the Greene-Plesser group give nothing new, using a homogeneous rescaling of the projective coordinates, e.g. for the factor generated by $g_3 = (1,0,0,-1,0)$ with $\lambda_3^4 = 1$ one finds $g_3 \sim g_1^3g_2^3$.

The labels (η, α, κ) are identified as $(\eta, \alpha, \kappa) \sim (\eta\lambda_1\lambda_2^{-1}, \alpha\lambda_1^3\lambda_2^3, \kappa)$ under the orbifold group. In the following we choose to label each orbit of curves by $(\alpha, \kappa) := (e^{i\pi/6}, \alpha, \kappa)$. Note that a rotation of η corresponds to a change of sign for α in this notation, $(e^{3i\pi/6}, \alpha, \kappa) = (-\alpha, \kappa)$. Instead of choosing a fixed η we can also fix the sign of α and keep two choices for η^3.

To calculate the domain wall tensions and the superpotentials for the vacua $C_{\alpha_1,\kappa}$ and $C_{\alpha_2,\kappa}$ we will study two families of divisors. The family $Q(\mathcal{D}_1) = x_2^6 + \hat{z}x_3^6$ interpolates between vacua related by a sign flip of η^3 or of the root α of the quartic equation. The family $Q(\mathcal{D}_2) = x_4^4 + \hat{z}x_1^6x_4^2$ interpolates between any two different roots α.

First divisor

We start with the analysis of the divisor

$$Q(\mathcal{D}_1) = x_2^6 + z_3x_3^6 \,. \tag{3.6}$$

To obtain some geometrical understanding of the surface defined by the intersection $P = 0 = Q(\mathcal{D}_1)$ we explicitly solve for $x_3 = (-z_3)^{-1/6}x_2$ and rescale x_2 to find

$$P_{\mathcal{D}_1} = x_1^{12} + x_2^6 + x_4^4 + x_5^3 + \tilde{\psi}x_1x_2^2x_4x_5 + \phi x_1^6x_4^2 \,. \tag{3.7}$$

Here $\tilde{\psi} = u_1^{-1/6}u_2^{-1/4}$, $\phi = u_2^{-1/2}$ are expressed in terms of the previous parameters as

$$u_1 = -\frac{z_1}{z_3}(1-z_3)^2, \qquad u_2 = z_2 \,. \tag{3.8}$$

Changing coordinates to $\tilde{x}_2 = x_2^2$ displays the family $\mathcal{D}_1$ as a double cover of a family of toric K3 surfaces associated to a GLSM with charges

$$\mathcal{D}_1 : \quad \begin{array}{cc|ccccc} & 0 & 1 & 2 & 4 & 5 & 6 \\ \hline \tilde{l}^1 & -6 & -1 & 2 & 0 & 2 & 3 \\ \tilde{l}^2 & 0 & 1 & 0 & 1 & 0 & -2 \end{array} \tag{3.9}$$

and with the two algebraic K3 moduli (3.8). The two covers are distinguished by a choice of sign for x_2.

The family of algebraic K3 manifolds obtained from (3.7) by the variable change $\tilde{x}_2 = x_2^2$ generically has four parameters with the two extra moduli multiplying the monomials $x_1^3x_4^3$ and $x_1^9x_4$. Since these terms are forbidden by the Greene-Plesser group of the Calabi-Yau threefold, the embedded surface is at a special symmetric point with the coefficients of these monomials set to zero. The periods on the K3 surface at this point can be computed from the GKZ system for the two parameter family, obtained from (2.21) with the charge vectors $\{\tilde{l}\}$ in eq. (3.9):

$$\begin{aligned} \mathcal{L}_1^{\mathcal{D}} &= \tilde{\theta}_1(2\tilde{\theta}_1 - 1)\prod_{k=0}^{2}(-3\tilde{\theta}_1 + 2\tilde{\theta}_2 + k) - \frac{9}{2}u_1(\tilde{\theta}_1 - \tilde{\theta}_2)\prod_{k=1,2,4,5}(6\tilde{\theta}_1 + k)\,, \\ \mathcal{L}_2^{\mathcal{D}} &= \tilde{\theta}_2(\tilde{\theta}_2 - \tilde{\theta}_1) - u_2(2\tilde{\theta}_2 - 3\tilde{\theta}_1)(2\tilde{\theta}_2 - 3\tilde{\theta}_1 + 1)\,, \end{aligned} \tag{3.10}$$

where $\tilde{\theta}_a = u_a \frac{d}{du_a}$. Apart from the regular solutions this system has two extra solutions depending on fractional powers in the u_i:

$$\begin{aligned}
\pi_1(u_1,u_2) &= \frac{c_1}{2}\, B_{\{\tilde{l}\}}(u_1,u_2;\tfrac{1}{2},0) = \frac{4c_1}{\pi}\sqrt{u_1}\ {}_2F_1(-\tfrac{1}{4},-\tfrac{3}{4},\tfrac{1}{2},4u_2) + \mathcal{O}(u_1^{3/2})\,,\\
\pi_2(u_1,u_2) &= \frac{c_2}{2}\, B_{\{\tilde{l}\}}(u_1,u_2;\tfrac{1}{2},\tfrac{1}{2}) = \frac{12c_2}{\pi}\sqrt{u_1u_2}\ {}_2F_1(-\tfrac{1}{4},\tfrac{1}{4},\tfrac{3}{2},4u_2) + \mathcal{O}(u_1^{3/2})\,. \qquad (3.11)
\end{aligned}$$

Here c_a are some normalization constants not determined by the differential operators. Later they will be fixed to one by studying the geometric periods on the surface.

As indicated, the exceptional solutions vanish at the critical point $u_1 = 0$ as $\sim \sqrt{u_1}$, with the coefficient a hypergeometric series in the modulus $u_2 = z_2$. These solutions arise as the specialization of the standard solutions of the four parameter family of K3 manifolds to the special symmetric point.[14] Since $u_1 = 0$ is not at the discriminant locus of the K3 family for general u_2, there is no geometric vanishing cycle associated with the zero of $\pi_{1,2}$. Instead the zero at $u_1 = 0$ arises from the 'accidental' cancellation between the volumes of different classes at the symmetric point.[15] The periods (3.11) have the special property that their leading terms $\sim \sqrt{u_1}$ near the critical point $u_1 = 0$ can be written in closed form as

$$\lim_{z_3\to 1} \frac{\pi_a(u_1,u_2)}{(1-z_3)} = \frac{4c_a}{\pi}\cdot \left.\frac{(i\alpha)(2\alpha^2-\phi)(\alpha^2+\phi)}{\psi^3}\right|_{\alpha=\alpha_{a,+}}, \qquad (3.12)$$

where

$$\alpha_{1,\pm} = \pm\sqrt{\frac{-\phi+\sqrt{\phi^2-4}}{2}}\,, \qquad \alpha_{2,\pm} = \pm\sqrt{\frac{-\phi-\sqrt{\phi^2-4}}{2}}\,, \qquad (3.13)$$

denote the roots of the quartic equation $\alpha^4+\phi\alpha^2+1=0$ appearing in the definition (3.5). Hence the leading part of the two K3 periods near the symmetric point is proportional to a rational function in the coefficients of the defining equations for the curve, evaluated at the critical points.

We will first compute the domain wall tensions by integrating the periods $\pi_{1,2}$ of the surface D_1. Note that the K3 periods π_a depend on $\xi = \sqrt{z_3}$ via their dependence on u_1 and the sign of the square root correlates with the sign of α. To obtain the off-shell tension, we integrate $\pi_a(\xi)$ as

$$\mathcal{T}_a^{(\pm)}(z_1,z_2,z_3) = \frac{1}{2\pi i}\int_{\xi_0}^{\pm\sqrt{z_3}} \pi_a(\xi)\,\frac{d\xi}{\xi}\,, \qquad (3.14)$$

where ξ_0 denotes a fixed reference point. For example, the period π_1 integrates to

$$\begin{aligned}
\frac{4\pi i\,\mathcal{T}_1^{(\pm)}}{c_1} &= \int_{\xi_0}^{\pm\sqrt{z_3}} \sum_{n_1,n_2\geq 0} \frac{\Gamma(4+6n_1)\left(-\frac{z_1}{\xi^2}(1-\xi^2)^2\right)^{n_1+\frac{1}{2}} z_2^{n_2}}{\Gamma(2+2n_1)^2\Gamma(1+n_2)\Gamma\left(\frac{1}{2}-n_1+n_2\right)\Gamma\left(\frac{5}{2}+3n_1-2n_2\right)}\frac{d\xi}{\xi}\\
&= \sum_{n_1,n_2\geq 0} \left.\frac{\Gamma(4+6n_1)(-z_1)^{n_1+\frac{1}{2}} z_2^{n_2}\left(\xi^2-1\right)^{2n_1+2}{}_2F_1\left(1,\frac{3}{2}+n_1,\frac{1}{2}-n_1,\xi^2\right)}{(1+2n_1)\Gamma(2+2n_1)^2\Gamma(1+n_2)\Gamma\left(\frac{1}{2}-n_1+n_2\right)\Gamma\left(\frac{5}{2}+3n_1-2n_2\right)\xi^{2n_1+1}}\right|_{\xi=\xi_0}^{\xi=\pm\sqrt{z_3}} \qquad (3.15)
\end{aligned}$$

[14] An explicit illustration of this fact is given in the case of the second family of divisors below.

[15] One parameter controlling the difference of these volumes is the direction of the off-shell modulus.

where the contribution from the reference point ξ_0 can be set to zero by choosing $\xi_0 = i$ as the lower bound. This will be used to split the result of the integral for the domain wall tension into two contributions of the superpotentials from the endpoints as in eq. (2.5). This split is not obvious in general, and ambiguous with respect to adding rational multiples of bulk periods. In the example we can use the $\mathbb{Z}_2$ symmetry acting on the curves to require that the superpotentials obey $\mathcal{W}_1^{(+)} = -\mathcal{W}_1^{(-)}$. With this convention and the particular choice of ξ_0 above, we obtain $\frac{1}{2\pi i}\int_{\xi_0}^{\pm\sqrt{z_3}} \pi_a(\xi)\,\frac{d\xi}{\xi} = \mathcal{W}_a^{(\pm)}$ or $\frac{1}{2\pi i}\int_{-\sqrt{z_3}}^{+\sqrt{z_3}} \pi_a(\xi)\,\frac{d\xi}{\xi} = \mathcal{W}_a^{(+)} - \mathcal{W}_a^{(-)} = 2\mathcal{W}_a^{(+)}$.

According to the discussion in sect. 2, the superpotentials $\mathcal{W}_a^{(\pm)}(z_1,z_2,z_3)$ restrict to the on-shell superpotentials $W_a^{(\pm)}(z_1,z_2)$ with vanishing derivative in the open-string direction z_3 at the critical point:

$$W_a^{(\pm)}(z_1,z_2) = \mathcal{W}_a^{(\pm)}\Big|_{z_3=1}, \qquad \xi\partial_\xi \mathcal{W}_a^{(\pm)}(z_1,z_2,\xi^2)\Big|_{z_3=1} = \pm\frac{1}{2\pi i}\;\pi_a|_{u_1=0} = 0\,. \tag{3.16}$$

For the above integrals one obtains

$$W_1^{(\pm)} = \mp\frac{c_1}{8\pi}\sum_{n_1,n_2\geq 0}\frac{(-1)^{n_1+1}\,\Gamma\left(-n_1-\frac{1}{2}\right)\Gamma\left(6n_1+4\right)\,z_1^{n_1+\frac{1}{2}}z_2^{n_2}}{\Gamma\left(n_1+\frac{3}{2}\right)\Gamma\left(2n_1+2\right)\Gamma\left(3n_1-2n_2+\frac{5}{2}\right)\Gamma\left(n_2+1\right)\Gamma\left(-n_1+n_2+\frac{1}{2}\right)}\,, \tag{3.17}$$

$$W_2^{(\pm)} = \mp\frac{c_2}{8\pi}\sum_{n_1,n_2\geq 0}\frac{(-1)^{n_1+1}\,\Gamma\left(-n_1-\frac{1}{2}\right)\Gamma\left(6n_1+4\right)\,z_1^{n_1+\frac{1}{2}}z_2^{n_2+\frac{1}{2}}}{\Gamma\left(n_1+\frac{3}{2}\right)\Gamma\left(2n_1+2\right)\Gamma\left(3n_1-2n_2+\frac{3}{2}\right)\Gamma\left(n_2+\frac{3}{2}\right)\Gamma\left(-n_1+n_2+1\right)}\,.$$

These functions can be expressed in terms of the bulk generating function as

$$W_1^{(\pm)} = \mp\frac{c_1}{8}\,B_{\{l\}}\left(z_1,z_2;\tfrac{1}{2},0\right)\,, \qquad W_2^{(\pm)} = \mp\frac{c_2}{8}\,B_{\{l\}}\left(z_1,z_2;\tfrac{1}{2},\tfrac{1}{2}\right)\,. \tag{3.18}$$

Complementary, the tensions $\mathcal{T}_a^{(\pm)}(z_1,z_2,z_3)$ and their on-shell restrictions $T_a^{(\pm)}(z_1,z_2)$ can be described as solutions to the large GKZ system for the relative cohomology problem derived in refs. [16, 17, 26]. For the family (3.6) the additional charge vector is

	0	1	2	3	4	5	6	7	8
l^3	0	0	1	-1	0	0	0	-1	1

.

Together with the charge vectors l^1 and l^2 for the Calabi-Yau hypersurface this defines the extended hypergeometric system of the form (2.21), which can be associated with a dual fourfold X_4 for a M/F-theory compactification [16, 20, 27]. For a description of X_4 as a toric hypersurface we refer to app. A.1. From the extended charge vectors one obtains after an appropriate factorization the system of differential operators[16]

$$\begin{aligned}
\mathcal{L}_1 &= (\theta_1+\theta_3)(\theta_1-\theta_3)(3\theta_1-2\theta_2) - 36z_1(6\theta_1+5)(6\theta_1+1)(\theta_2-\theta_1+2z_2(1+6\theta_1-2\theta_2))\,,\\
\mathcal{L}_2 &= \theta_2(\theta_2-\theta_1) - z_2(3\theta_1-2\theta_2-1)(3\theta_1-2\theta_2)\,,\\
\mathcal{L}_3 &= \theta_3(\theta_1+\theta_3) + z_3\theta_3(\theta_1-\theta_3)\,.
\end{aligned} \tag{3.19}$$

[16] The first operator is obtained after a factorization similar to the one described in ref. [11] for the underlying threefold.

After a simple variable transformation $y = \ln(z_3)$, with the variable y centered at the critical point, the solutions to this system describe the expansion of the periods on the relative homology $H^3(Z^*, \mathcal{D}_1)$ around the critical point. These include the off-shell tensions $\mathcal{T}_a^{(\pm)}(z_1, z_2, z_3)$ (3.14), which restrict to the functions (3.18), and in addition the closed-string periods $\Pi(z_1, z_2)$. The integration from the geometric surface periods of the subsystem fixes the z_3-dependent piece. The GKZ system restricts the afore mentioned integration constant to a linear combination of the closed-string periods $\Pi(z_1, z_2)$. The rational coefficients appearing in this combination can be determined by a monodromy argument, as in ref. [12] and as exemplified for a non-compact limit of the Calabi-Yau threefold in sect. A.2.

Finally one may also characterize the critical tensions $T_a^{(\pm)}$, or, for the above reasons also the critical superpotentials $W_a^{(\pm)}$, as the solution to the inhomogeneous Picard-Fuchs equation (2.15), which makes contact to the normal function approach of [13]. Due to

$$\mathcal{L}_1 = \mathcal{L}_1^{bulk}(\theta_1, \theta_2) - (3\theta_1 - 2\theta_2)\theta_3^2, \qquad \mathcal{L}_2 = \mathcal{L}_2^{bulk}(\theta_1, \theta_2), \tag{3.20}$$

we observe that only the first operator may acquire a non-zero inhomogeneous term at the critical point. This term is determined by the leading behavior of the surface periods π_a in the limit $u_1 \to 0$. Acting with $\mathcal{L}_1^{bdry} = (3\theta_1 - 2\theta_2)\theta_3$ on the terms on the right hand side of eqs. (3.11) one obtains the inhomogeneous Picard-Fuchs equations

$$\begin{aligned} \mathcal{L}_1^{bulk} W_1^{(\pm)} &= \mp \frac{3c_1}{2\pi^2} \sqrt{z_1}\ {}_2F_1(\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{2}, 4z_2) = f_1(\alpha_{1,\pm}) \ , \\ \mathcal{L}_1^{bulk} W_2^{(\pm)} &= \mp \frac{3c_2}{2\pi^2} \sqrt{z_1 z_2}\ {}_2F_1(\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{3}{2}, 4z_2) = f_1(\alpha_{2,\pm}) \ , \end{aligned} \tag{3.21}$$

while $\mathcal{L}_2^{bulk} W_a^{(\pm)} = 0$. The roots (3.13) of the quartic equation are identified with the label $(a, \pm)$ of the curves in the right hand side of eq. (3.21). Indeed, as a consequence of eq. (3.12), the inhomogeneous terms can again be written in closed form as

$$\mathcal{L}_a^{bulk} W(\alpha) = f_a(z, \alpha) \, ,$$

with $W(\alpha_{a,\pm}) = W_a^{(\pm)}$ and the $f_a(z, \alpha)$ rational functions in the coefficients of the defining equation:

$$f_1(z, \alpha) = \frac{3c}{2\pi^2} \cdot \frac{i\,\phi\,\alpha(\alpha^2 + \phi)}{\psi^3} \, , \qquad f_2(z, \alpha) = 0 \, , \tag{3.22}$$

for $c = c_1 = c_2$. As is apparent from (3.21), this function satisfies a hypergeometric equation $\mathcal{L}^{inh} f_1 = 0$. The hypergeometric operator is related to the surface operators by eq. (2.18). In the present case, the relevant operator arises from $\mathcal{L}_2^{\mathcal{D}}$, that is $\mathcal{L}^{inh} = (\mathcal{L}_2^{\mathcal{D}} + [\mathcal{L}_1^{bdry}, \mathcal{L}_2^{\mathcal{D}}]\mathcal{L}_1^{bdry\,-1})|_{\hat{z}_{crit}}$, while $\mathcal{L}_1^{\mathcal{D}}$ becomes irrelevant. With

$$\mathcal{L}_2^{\mathcal{D}}|_{\hat{z}_{crit}} = \theta_2(\theta_2 - \tfrac{1}{2}) - 4z_2(\theta_2 - \tfrac{1}{4})(\theta_2 - \tfrac{3}{4}) \, , \qquad \mathcal{L}_1^{bdry}|_{\hat{z}_{crit}} = i(\theta_2 - \tfrac{3}{4}) \, ,$$

one obtains

$$\mathcal{L}^{inh} = \theta_2(\theta_2 - \tfrac{1}{2}) - 4z_2(\theta_2 - \tfrac{1}{4})(\theta_2 + \tfrac{1}{4}) \, . \tag{3.23}$$

In the above we have used that the relevant surface period is the solution to the Picard-Fuchs system $\{\mathcal{L}_b^{\mathcal{D}}\}$ with index $\frac{1}{2}$ in the variable u_1 to set $\tilde{\theta}_1 = \frac{1}{2}$.

A-model expansion

By mirror symmetry, these functions should have an integral instanton expansion when expressed in terms of the appropriate coordinates and taking into appropriately the contributions from multi-covers [5]. For the critical branes at fixed $\hat{z}$, we use the modified multi-cover formulae of the type proposed in refs. [12, 14, 35]:

$$\frac{W_1^{(\pm)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2} \sum_{k\,odd} \sum_{\substack{d_1\,odd \\ d_2 \geq 0}} n_{d_1,d_2}^{(1,\pm)} \frac{q_1^{kd_1/2}\, q_2^{kd_2}}{k^2}\,, \tag{3.24}$$

$$\frac{W_2^{(\pm)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2} \sum_{k\,odd} \sum_{\substack{d_1\,odd \\ d_2\,odd}} n_{d_1,d_2}^{(2,\pm)} \frac{q_1^{kd_1/2}\, q_2^{kd_2/2}}{k^2}\,. \tag{3.25}$$

In this way one obtains the integer invariants in Tab. 1 for $c_a = 1$. As can be guessed from these numbers, the superpotentials for $a = 1, 2$ are in fact not independent, but related by a $\mathbb{Z}_2$ symmetry. The family of Calabi-Yau hypersurfaces (3.2) develops a singularity at the discriminant locus $\Delta = 1 - 4z_2 = 0$, which is mirror to a curve of A_1 singularities [46, 47]. On the B model side the $\mathbb{Z}_2$ monodromy around the singular locus $\Delta = 0$ exchanges the two sets of roots $\alpha_{1,\pm}$ and $\alpha_{2,\pm}$ in eq. (3.13). Accordingly, the superpotentials $W_1^{(\pm)}$ and $W_2^{(\pm)}$ are also exchanged as can be seen from the structure of the inhomogeneous terms. On the level of periods this monodromy action yields

$$t_1 \to t_1 + 3t_2, \qquad t_2 \to -t_2\,. \tag{3.26}$$

As a result the invariants of W_2 are related to that of W_1 by the $\mathbb{Z}_2$ quantum symmetry $q_1 \to q_1 q_2^3,\ q_2 \to q_2^{-1}$ generated by (3.26).[17]

Extremal transition and a non-compact limit

The above results and the normalization obtained by integration from the subsystem can be verified by taking two different one parameter limits. At the singular locus $\Delta = 0$, there is an extremal transition to the one parameter family mirror to a degree (6,4) complete intersection hypersurface in $\mathbb{P}_{1,1,1,2,2,3}$. From eq. (3.26) it follows that the transition takes place at $q_2 = 1$, predicting the relation

$$\sum_{\ell=0}^{3k} n_{k,\ell}^{(a,+)}(\mathbb{P}_{1,2,2,3,4}[12]) = n_k(\mathbb{P}_{1,1,1,2,2,3}[6,4])\,, \quad a = 1, 2\,, \tag{3.27}$$

where (k, ℓ) denote the degree in q_1 and q_2, respectively. The finiteness of the sum over ℓ follows from the symmetry (3.26). From the left hand side of the above equation one gets

$$n_k = 64,\ 48\,576,\ 265\,772\,480,\ 2\,212\,892\,036\,032,\ 22\,597\,412\,764\,939\,776,\ \ldots \tag{3.28}$$

[17]The $\mathbb{Z}_2$ symmetry is also realized on the closed-string invariants, see the results of ref. [11].

$n^{(1,+)}_{d_1,d_2}$

$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5
1	16	48	0	0	0	0
3	-432	-480	38688	10800	0	0
5	45440	-78192	5472	92812032	146742768	26162880
7	-7212912	25141920	-165384288	61652832	327357559584	1094178697056
9	1393829856	-6895024080	49628432160	-426927933792	261880092960	1383243224519472
11	-302514737008	1905539945472	-14487202588320	131586789107520	-1448971951799232	1383991826496480
13	70891369116256	-538859226100800	4335978084777792	-39691782337561536	440278250387930640	-5799613460160838608
15	-17542233743427360	155713098595732704	-1328641212531217728	12308540119113753936	-132576278776141577664	1710971659352271824160

$n^{(2,+)}_{d_1,d_2}$

$q_1^{1/2} \backslash q_2^{1/2}$	1	3	5	7	9	11	13
1	48	16	0	0	0	0	0
3	0	10800	38688	-480	-432	0	0
5	0	82080	26162880	146742768	92812032	5472	-78192
7	0	-10780160	241323840	88380335472	702830702688	1094178697056	327357559584
9	0	1843890480	-36172116480	932346639840	364829042312640	3751178206812144	*
11	0	-369032481792	6979488962400	-143329914498240	4246347124847520	*	*

Table 1: Disc invariants for the on-shell superpotentials $W_a^{(+)}$ of the threefold $\mathbb{P}_{1,2,2,3,4}[12]$.

for the first invariants of $\mathbb{P}_{1,1,1,2,2,3}[6,4]$. This can be checked by a computation for the complete intersection manifold with the inhomogeneous Picard-Fuchs equation

$$\mathcal{L}\ W(z) = \frac{4\sqrt{z}}{(2\pi i)^2}\ , \qquad \mathcal{L} = \theta^4 - 48z(6\theta+5)(6\theta+1)(4\theta+3)(4\theta+1)\ . \tag{3.29}$$

Another interesting one modulus limit is obtained for $z_2 \to 0$, where X degenerates to the non-compact hypersurface

$$X^\flat:\quad y_1^2 + y_2^3 + y_3^6 + y_4^6 + y_5^{-6} + \hat{\psi}\, y_1 y_2 y_3 y_4 y_5 = 0, \qquad \hat{\psi} = \frac{\psi}{\sqrt{\phi}} = z_1^{-1/6} \tag{3.30}$$

in weighted projective space $\mathbb{P}^4_{3,2,1,1,-1}$, with the new variables y_i related to the x_i by

$$y_1 = \phi^{1/2} x_4 x_1^3,\ y_2 = x_5,\ y_3 = x_2,\ y_4 = x_3,\ y_5 = x_1^{-2}\ .$$

The non-compact 3-fold $X^\flat$ is a local model for a certain type of singularity associated with the appearance of non-critical strings and has been studied in detail in ref. [48].

In this limit the curves $C_{\alpha_{2,\pm},\kappa}$ of eq. (3.5) are pushed to the boundary of the local threefold geometry $X^\flat$ and the domain wall tension between $C_{\alpha_{2,+},\kappa}$ and $C_{\alpha_{2,-},\kappa}$ becomes independent of the modulus z_1, which is reflected by the fact that all the disc invariants of W_2 vanish in the limit $z_2 \to 0$. The curves $C_{\alpha_{1,\pm},\kappa}$ become

$$C^\flat_{\varepsilon,\kappa} = \left\{ y_3 = \eta\, y_4\ ,\ y_1 y_5^3 = \varepsilon\ ,\ y_2 y_5 = \kappa\, y_4 \sqrt{\varepsilon \eta \hat{\psi}} \right\}\ , \quad \varepsilon = \pm i\ ,\ \kappa = \pm i\ , \tag{3.31}$$

where $\varepsilon = \pm i$ distinguishes between the two roots $\alpha_{1,+}$ and $\alpha_{1,-}$. In app. A.2 we show, that the 3-chain integral representing the domain wall tension in $X^\flat$ descends to an Abel-Jacobi map on a Riemann surface, which can be computed explicitly as an geometric integral. The invariants $n^{[6]}$ obtained for the superpotential in the non-compact geometry $X^\flat$ are reported in app. A.2 and they agree with the q_2^0 term of T_1, $n_{k,0} = n_k^{[6]}$.

A second family of divisors and symmetric K3s

The same critical points can be embedded into a different family of divisors

$$Q(\mathcal{D}_2) = x_4^4 + z_3 z_2^{-1/2} x_1^6 x_4^2 \,. \tag{3.32}$$

Our motivation to consider this second family in detail is two-fold. Firstly, the Hodge problem on the surface is equivalent to that of a two parameter family of K3 surfaces at a special point in the moduli, which can be studied explicitly without too many technicalities. We will explicitly show that the relevant zero of the period vector arises at an orbifold point of the K3, which has been interpreted as a point with a half-integral B-field for the closed-string compactification on the local geometry [49]. Secondly, this family tests a different direction of the off-shell deformation space of the brane, leading to a different off-shell superpotential $\mathcal{W}$ for the deformation (3.32). However, since the family contains the curves $C_{\alpha,\kappa}$ for $z_3 = -\alpha^2 z_2^{1/2}$, the critical superpotential has to be the same as the one obtained for the family $\mathcal{D}_1$ in eq. (3.18). The agreement with the previous result and normalization gives an explicit illustration of the fact that different parametrizations of the off-shell directions, corresponding to a different choice of light fields represented by different relative cohomologies, fit together consistently near the critical locus.

As the critical point is determined by the vanishing condition (2.7), we again study the subsystem $P = Q(\mathcal{D}_2) = 0$. Solving for x_4 and changing coordinates to $\tilde{x}_1 = x_1^4$, the surface can be described as a cover[18] of a mirror family of K3 hypersurfaces

$$\tilde{x}_1^3 + x_2^6 + x_3^6 + x_5^3 + \tilde{\psi}\tilde{x}_1 x_2 x_3 x_5 + \tilde{\phi}(x_2 x_3)^3 = 0 \,.$$

Here $\tilde{\psi}^{-6} := u = -\frac{z_1 z_2}{z_3^3}(z_2 - z_3 + z_3^2)^2$ and the parameter $\tilde{\phi}$ is zero for the embedded surface. At $\tilde{\phi} = 0$, the GLSM for this family is defined by the charges

$$\begin{array}{cc|cccc} & 0 & 1 & 2 & 3 & 5 \\ \hline \tilde{l} & -6 & 2 & 1 & 1 & 2 \end{array} \,.$$

The GKZ system for this one modulus GLSM has an exceptional solution

$$\pi(u) = \frac{c}{2}\, B_{\{\tilde{l}\}}(u; \tfrac{1}{2}) = \frac{c}{2}\sum_{n=0}^{\infty} \frac{\Gamma(4+6n)}{\Gamma(2+2n)^2 \Gamma(\frac{3}{2}+n)^2} u^{n+\frac{1}{2}} \,, \tag{3.33}$$

that vanishes at the critical point $u = 0$. To get a better understanding of this solution and of the integral periods on the surface, one may describe π as a regular solution of the two parameter family of K3 surfaces parametrized by $\tilde{\psi}$ and $\tilde{\phi}$, restricted to the symmetric point $\tilde{\phi} = 0$. The charges of the GLSM for the two parameter family of K3 manifolds are

$$\begin{array}{cc|ccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \tilde{l}^1 & -3 & 1 & 0 & 0 & 1 & 1 \\ \tilde{l}^2 & 0 & 0 & 1 & 1 & 0 & -2 \end{array} \,.$$

[18] The change from x_1 to $\tilde{x}_1$ gives a fourfold cover acted on by a remaining $\mathbb{Z}_2$ action generated by g_1 in (3.4).

The two algebraic moduli of this family are $v_1 = -\tilde{\phi}\tilde{\psi}^{-3}$ and $v_2 = \tilde{\phi}^{-2}$ and these are related to the single modulus of the embedded surface by $u = \tilde{\psi}^{-6} = v_1^2 v_2$. The principal discriminant locus for this family has the two components

$$\Delta = \Delta_0 \cdot \Delta_1 = (1 + 54v_1 + 729v_1^2 - 2916v_1^2 v_2) \cdot (1 - 4v_2) \ .$$

The periods near $\tilde{\phi} = 0$ can be computed in the phase of the two parameter GLSM with coordinates $u_1 = v_1 v_2^{1/2}$ and $u_2 = v_2^{-1/2}$. The hypergeometric series

$$\tilde{\pi}(u_1, u_2) = \frac{c}{2\pi^2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{\Gamma(1+3n)\Gamma(\frac{1}{2} - n + p)^2}{\Gamma(1+n)^2 \Gamma(2 - n + 2p)} u_1^n u_2^{1+2p-n} \tag{3.34}$$

is a solution of the Picard-Fuchs equation that restricts to $\pi(\sqrt{u})$ in the limit $u_2 = 0$. This series can be expressed with the help of a Barnes type integral as

$$\tilde{\pi}(u_1, u_2) = -\frac{c}{2\pi^2} \int_{\mathcal{C}_+} \sum_{n=0}^{\infty} \frac{\Gamma(1+3n)\Gamma(\frac{1}{2} + s)^2 \Gamma(1+s)\Gamma(-s)(-1)^s}{\Gamma(1+n)^2 \Gamma(2 + n + 2s)} (u_1 u_2)^n u_2^{1+2s} \tag{3.35}$$

$$+ \frac{c}{2\pi^2} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\Gamma(1+3n)\Gamma(\frac{1}{2} - p)^2}{\Gamma(1+n)^2 \Gamma(2 + n - 2p)} (u_1 u_2)^n u_2^{1-2p} \ , \tag{3.36}$$

where the contour $\mathcal{C}_+$ encloses the poles of the Gamma functions on the positive real line including zero. To relate the special solution $\tilde{\pi}(u_1, u_2)$ to the integral periods on the K3, one may analytically continue it to large complex structure by closing the contour to the left and obtains

$$\begin{aligned} \tilde{\pi}(v_1, v_2) &= \frac{c}{2\pi i} \sum_{n,p=0}^{\infty} \frac{\Gamma(1+3n) v_1^n v_2^p \left(-i\pi + \ln(v_2) + 2(\Psi(1+n-2p) - \Psi(1+p))\right)}{\Gamma(1+n)^2 \Gamma(1+n-2p)\Gamma(1+p)^2} \\ &= c\,\omega_0 \left(t_2^{K3} - \frac{1}{2} \right) \ . \end{aligned} \tag{3.37}$$

Here $\omega_0 = B_{\tilde{l}}(v_a; 0, 0)$ is the fundamental integral period at large volume, and $t_2^{K3} = (2\pi i \omega_0)^{-1} \partial_{\rho_2} B_{\tilde{l}}|_{\rho_a = 0}$ is the integral period associated with the volume of another 2-cycle C, which is mirror to the base of the elliptic fibration defined by the GLSM of the A model side.

From the last expression it follows that the zero of the K3 period vector associated with the D-brane vacuum arises at the locus

$$J^{K3} = \mathrm{Im}\, t_2^{K3} = 0 \ , \qquad B^{K3} = \mathrm{Re}\, t_2^{K3} = \tfrac{1}{2} \ , \tag{3.38}$$

which, in the closed string compactification on this local K3 geometry, is interpreted as a 2-cycle of zero volume with a half-integral B-field. Indeed, in the limit $u = 0 = u_1$, eq. (3.34) becomes

$$\tilde{\pi}(u_1, u_2)|_{u_1 = 0} \sim \ln\left(\frac{1 - 2v_2 - \sqrt{1 - 4v_2}}{2v_2} \right) - i\pi \ ,$$

expanded around $v_2 = \infty$. The first term on the right hand side is the period for the compact cycle of the $\mathbb{C}^2/\mathbb{Z}_2$-quotient singularity studied in ref. [49], which is zero on the discriminant locus $\Delta_1 = 0$, but a constant at $v_2 = \infty$. The zero associated with the critical point hence does not appear on the principal discriminant, but at an orbifold point with non-vanishing complex quantum volume. It has been argued in refs. [12, 13], that the A model data associated with the critical points of the present type include $\mathbb{Z}_2$-valued open-string degrees of freedom from the choice of a discrete gauge field on the A-brane. Here we see that to this discrete choice in the A model there corresponds, at least formally, a half-integral valued B-field for the tension in the B-model geometry. It would be interesting to study this phenomenon and its $\mathbb{C}^2/\mathbb{Z}_n$ generalizations in more detail, and we hope to come back to this issue elsewhere.

As in the previous parametrization, the tensions can be computed from the integrals

$$T_a = \frac{1}{2\pi i}\int_*^{\beta_a} \pi(u(\xi))\frac{d\xi}{\xi}\ ,$$

where $\beta_{1/2} = \pm i z_2^{1/4}\alpha_{1/2}$, with $\alpha_{1/2}$ defined in eq. (3.13). We again choose the reference point such that $W^{(+,\alpha)} = -W^{(-,\alpha)}$ and find

$$W^{(\pm,\alpha_1)} = \mp\frac{c}{8}\cdot B_{\{l^1,l^2\}}\left(z_1, z_2; \tfrac{1}{2}, 0\right)\ ,\quad W^{(\pm,\alpha_2)} = \mp\frac{c}{8}\cdot B_{\{l^1,l^2\}}\left(z_1, z_2; \tfrac{1}{2}, \tfrac{1}{2}\right)\ , \tag{3.39}$$

which is in agreement with (3.18) for $c = 1$.

3.2 Degree 14 hypersurface in $\mathbb{P}_{1,2,2,2,7}$

The charge vectors of the GLSM for the A model manifold are given by [11]

$$\begin{array}{cc|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline l^1 & -7 & -3 & 1 & 1 & 1 & 0 & 7 \\ l^2 & 0 & 1 & 0 & 0 & 0 & 1 & -2 \end{array}\ .$$

The hypersurface constraint for the mirror manifold, written in homogeneous coordinates in $\mathbb{P}_{1,2,2,2,7}$ as well, is

$$P = x_1^{14} + x_2^7 + x_3^7 + x_4^7 + x_5^2 - \psi\, x_1x_2x_3x_4x_5 + \phi\, x_1^7 x_5\,, \tag{3.40}$$

where $\psi = z_1^{-1/7} z_2^{-1/2}$ and $\phi = z_2^{-1/2}$. The orbifold group acts as $x_i \to \lambda_k^{g_k^i} x_i$ with $\lambda_k^7 = 1$ and weights

$$\mathbb{Z}_7:\ g_1 = (1,-1,0,0,0),\quad \mathbb{Z}_7:\ g_2 = (1,0,-1,0,0),\quad \mathbb{Z}_7:\ g_3 = (1,0,0,-1,0)\,. \tag{3.41}$$

In this geometry we consider the set of curves

$$\begin{aligned} C_{\alpha,\pm} = \{x_3 = \eta x_4\,,\ x_5 = \alpha x_1^7\,,\ x_2^3 = \pm\sqrt{\alpha\eta\psi}\, x_4 x_1^4\}\,,& \\ \eta^7 = -1\,, \qquad \alpha^2 + \phi\alpha + 1 = 0\,,& \end{aligned} \tag{3.42}$$

with the following identification under the orbifold group: $(\eta, \alpha, \pm) \sim (\eta\lambda_2\lambda_3^{-1}, \alpha, \pm)$. By choosing representatives we can fix η completely and label the orbits by $(\alpha, \pm)$.

First divisor

The family of divisors

$$Q(\mathcal{D}_1) = x_3^7 + z_3 x_4^7 \tag{3.43}$$

contains the curves $C_{\alpha,\pm}$ for the critical value $z_3 = 1$. The periods on the family of surfaces is captured by the GLSM with charges

$$\begin{array}{cc|ccccc} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 5} & {\scriptstyle 6} \\ \hline \tilde{l}^1 & -7 & -3 & 1 & 2 & 0 & 7 \\ \tilde{l}^2 & 0 & 1 & 0 & 0 & 1 & -2 \end{array} .$$

with two algebraic moduli $u_1 = -\frac{z_1}{z_3}(1-z_3)^2$ and $u_2 = z_2$. The exceptional solutions

$$\begin{aligned} \pi_1 &= \frac{c_1}{2} B_{\{\tilde{l}\}}(u_1, u_2, \tfrac{1}{2}, 0) = -\frac{c_1}{2\pi}\sqrt{u_1}\ {}_2F_1(-\tfrac{7}{4}, -\tfrac{5}{4}, -\tfrac{1}{2}, 4u_2) + \mathcal{O}(u_1^{3/2}) , \\ \pi_2 &= \frac{c_2}{2} B_{\{\tilde{l}\}}(u_1, u_2, \tfrac{1}{2}, \tfrac{1}{2}) = \frac{35 c_2}{2\pi}\sqrt{u_1} u_2^{3/2}\ {}_2F_1(-\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{5}{2}, 4u_2) + \mathcal{O}(u_1^{3/2}) , \end{aligned} \tag{3.44}$$

vanish at the critical point $u_1 = 0$. Note that these are series in $\sqrt{z_3}$ and the sign of the root distinguishes the two different holomorphic curves $C_{\alpha,+}$ and $C_{\alpha,-}$ in (3.42). The superpotentials obtained from integrals similar to (3.14) are

$$\begin{aligned} W_1^{(\pm)} &= \pm \frac{c_1}{8} \sum_{n_i \geq 0} \frac{\Gamma\left(7n_1 + \frac{9}{2}\right) z_1^{n_1+\frac{1}{2}} z_2^{n_2}}{\Gamma\left(n_1+\frac{3}{2}\right)^3 \Gamma\left(7n_1 - 2n_2 + \frac{9}{2}\right) \Gamma\left(n_2+1\right) \Gamma\left(n_2 - 3n_1 - \frac{1}{2}\right)} , \\ W_2^{(\pm)} &= \pm \frac{c_2}{8} \sum_{n_i \geq 0} \frac{\Gamma\left(7n_1 + \frac{9}{2}\right) z_1^{n_1+\frac{1}{2}} z_2^{n_2+\frac{1}{2}}}{\Gamma\left(n_1+\frac{3}{2}\right)^3 \Gamma\left(7n_1 - 2n_2 + \frac{7}{2}\right) \Gamma\left(n_2+\frac{3}{2}\right) \Gamma\left(n_2 - 3n_1\right)} . \end{aligned} \tag{3.45}$$

They can be expressed in terms of the bulk generating function as

$$W_1^{(\pm)} = \pm \frac{c_1}{8} B_{\{l^1,l^2\}}\left(z_1, z_2; \tfrac{1}{2}, 0\right) , \qquad W_2^{(\pm)} = \pm \frac{c_2}{8} B_{\{l^1,l^2\}}\left(z_1, z_2; \tfrac{1}{2}, \tfrac{1}{2}\right) . \tag{3.46}$$

As in the previous example, these functions are the restrictions to the critical point $z_3 = 1$ of the off-shell tensions, which can be obtained as the solutions to the large GKZ system (2.21) of the relative cohomology problem derived in refs. [16, 17, 26]. For the family (3.43), the additional charge vector is

$$\begin{array}{cc|cccccccc} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle 6} & {\scriptstyle 7} & {\scriptstyle 8} \\ \hline l^3 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{array} .$$

This leads to the generalized hypergeometric system

$$\begin{aligned} \tilde{\mathcal{L}}_1 &= (\theta_1+\theta_3)(\theta_1-\theta_3)\left(7\theta_1 - 2\theta_2\right) - 7z_1\left(z_2\left(28\theta_1 - 4\theta_2 + 18\right) - 3\theta_1 + \theta_2 - 2\right) \times \\ &\qquad \times \left(z_2\left(28\theta_1 - 4\theta_2 + 10\right) - 3\theta_1 + \theta_2 - 1\right)\left(z_2\left(28\theta_1 - 4\theta_2 + 2\right) - 3\theta_1 + \theta_2\right) , \\ \tilde{\mathcal{L}}_2 &= \theta_2(\theta_2 - 3\theta_1) - z_2(7\theta_1 - 2\theta_2 - 1)(7\theta_1 - 2\theta_2) , \\ \tilde{\mathcal{L}}_3 &= \theta_3(\theta_1+\theta_3) + z_3\theta_3(\theta_1 - \theta_3) , \end{aligned} \tag{3.47}$$

annihilating the relative period integrals on the relative cohomology $H^3(Z^*, D_1)$ near the critical locus $y = \ln(z_3) = 0$. Again this system has an alternative origin as the GKZ system associated to an F-theory compactification on a dual 4-fold described in app. A.1.

Alternatively, one may characterize the normal functions as solutions to an inhomogeneous Picard-Fuchs equation. From

$$\tilde{\mathcal{L}}_1 = \mathcal{L}_1^{bulk} - (7\theta_1 - 2\theta_2)\theta_3^2, \qquad \tilde{\mathcal{L}}_2 = \mathcal{L}_2^{bulk},$$

one sees that only the first operator acquires an inhomogeneous term, which is determined by the leading part of the surface periods π_a. Acting with $(7\theta_1 - 2\theta_2)\theta_3$ on the terms in (3.44) one obtains the inhomogeneous Picard-Fuchs equations

$$\begin{aligned} \mathcal{L}_1^{bulk} W_1^{(\pm)} &= \mp \frac{7c_1}{16\pi^2}\sqrt{z_1}\, {}_2F_1\left(-\tfrac{3}{4}, -\tfrac{5}{4}, -\tfrac{1}{2}, 4z_2\right) \; = \pm c_1 f_1(\alpha_1) \ , \\ \mathcal{L}_1^{bulk} W_2^{(\pm)} &= \pm \frac{35c_2}{16\pi^2}\, z_1^{1/2} z_2^{3/2}\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{5}{2}, 4z_2\right) \; = \pm c_2 f_1(\alpha_2) \ . \end{aligned} \tag{3.48}$$

The inhomogeneous terms can be summarized as

$$f_1(\alpha) = -\frac{7i}{16\pi^2} \cdot \frac{\phi(\alpha+\phi)(6\alpha+\phi)}{\alpha^{1/2}\psi^{7/2}} \,, \tag{3.49}$$

where

$$\alpha_{1/2} = \frac{1}{2}\left(-\phi \pm \sqrt{\phi^2 - 4}\right) \,, \tag{3.50}$$

denote the roots of the quadratic equation in the defining equation (3.42).

A-model expansion

The superpotential $W_1^{(+)}$ is associated with the curve $C_{\alpha_1,+}$ and similarly $W_2^{(+)}$ with $C_{\alpha_2,+}$. With the normalization $c_1 = c_2 = 1$ and the multi-cover formulae (3.24) and (3.25), we obtain the integer invariants in Tab. 2. Similarly as in the previous example, the two superpotentials are related by a $\mathbb{Z}_2$ symmetry arising from the monodromy associated with an A_1 curve singularity [46, 47]. On the B-model side, the $\mathbb{Z}_2$ monodromy around the singular locus $\Delta = 0$ acts on the periods as $t_1 \to t_1 + 7t_2$, $t_2 \to -t_2$. The invariants of W_2 are related to that of W_1 by the $\mathbb{Z}_2$ quantum symmetry $q_1 \to q_1 q_2^7$, $q_2 \to q_2^{-1}$ induced by this monodromy.

Extremal transition and a non-compact limit

The above results and the normalization obtained by integration from the subsystem can be further verified by taking two different one parameter limits. At the singular locus $\Delta = 0$, there is an extremal transition to the one parameter family mirror to a degree eight hypersurface in $\mathbb{P}_{1,1,1,1,4}$ [50]. To study this transition, we rewrite the hypersurface constraint (3.40) as

$$P = \big(-\alpha\psi x_1^8 x_2 x_3 x_4 + x_2^7\big) + \big(x_3^7 + x_4^7\big) + \big(x_5 - \psi x_1 x_2 x_3 x_4 + (\alpha+\phi) x_1^7\big)\big(x_5 - \alpha x_1^7\big) \ . \tag{3.51}$$

$\frac{1}{2} \cdot n^{(1,+)}_{d_1,d_2}$

$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5	6	7
1	1	-14	-35	0	0	0	0	0
3	-1	14	-56	-126	-3416	-42182	-19481	-396
5	5	-126	1351	-8358	41643	-157990	87339	-27425384
7	-42	1414	-21455	195790	-1271585	6722898	-30564891	152513340
9	429	-18200	357070	-4322640	37056327	-248175368	1390770059	-7006648980
11	-4939	252854	-6077729	91502334	-980198345	8110498760	-55066462542	322702120822
13	61555	-3691114	104989899	-1889415220	24334523486	-241697136212	1953204386721	-13402394296330

$\frac{1}{2} \cdot n^{(2,+)}_{d_1,d_2}$

$q_1^{1/2} \backslash q_2^{1/2}$	1	3	5	7	9	11
1	0	-35	-14	1	0	0
3	0	0	28	-396	-19481	-42182
5	0	0	-70	1582	-16212	179144
7	0	0	448	-13804	195552	-1907430
9	0	0	-4004	157525	-2892204	34409872

Table 2: Disc invariants for the on-shell superpotentials $W_a^{(+)}$ of the threefold $\mathbb{P}_{1,2,2,2,7}[14]$.

The three summands indicated by the brackets vanish individually on the curves $C_{\alpha,\pm}$. At the singular locus $\phi = \pm 2$, the map to the hypersurface in $\mathbb{P}_{1,1,1,1,4}$ is provided by the identifications

$$x_1^8 x_2 x_3 x_4 = y_1^8, \quad x_2^7 = y_2^8, \quad x_3^7 = y_3^8, \quad x_4^7 = y_4^8, \quad x_5 \pm x_1^7 = y_5 \ ,$$

and this maps the curves $C_{\alpha,\pm}$ to the curves $C_{\zeta\mu}$ of ref. [33] in $\mathbb{P}_{1,1,1,1,4}[8]$.[19]

From the symmetry $t_2 \to -t_2$ it follows that the transition takes place at $q_2 = 1$, predicting the relation

$$\sum_{i=0}^{7k} n_{k,i}^{(a,+)}(\mathbb{P}_{1,2,2,2,7}[14]) = n_k(\mathbb{P}_{1,1,1,1,4}[8])\,, \tag{3.52}$$

where (k,i) denote the degree in q_1 and q_2, respectively. From the left hand side of the above equation one gets from the above tables

$$-\frac{1}{2}\, n_k = \ 48,\ 65616,\ 919252560,\ ...$$

for the invariants of $\mathbb{P}_{1,1,1,1,4}[8]$. This is in agreement with the results of [33, 34], up to a sign, which is convention.

On the other hand, the the q_2^0 term of the superpotential W_1 reproduces the invariants of the superpotential in the non-compact geometry $\mathcal{O}(-3)_{\mathbb{P}^2}$ studied in ref. [32], $n_{k,0} = n_k^{[3]}$ of Tab. 10 in app. A.2. To recover this limit geometrically from eq. (3.51) we define

$$\begin{aligned} &y_0 = -\alpha\psi x_1^8 x_2 x_3 x_4, \quad y_1 = x_2^7, \quad y_2 = x_3^7, \quad y_3 = x_4^7, \\ &x = \frac{x_5}{\phi} - \frac{\psi}{\phi} x_1 x_2 x_3 x_4 + \left(\frac{\alpha}{\phi} + 1\right) x_1^7\,, \quad z = \phi x_5 - \alpha\phi x_1^7\,, \end{aligned}$$

to write the hypersurface constraint as $P = (y_0 + y_1) + (y_2 + y_3) + xz$. The two roots $\alpha_{1/2}$ behave in the limit as $\alpha_{1/2} \sim -\phi^{\mp 1}$. Choosing $\alpha = \alpha_1$ in (3.51) and rescaling $x_5 \to \frac{x_5}{\phi}$, one finds

$$x = z_1^{-1/7} x_1 x_2 x_3 x_4 + x_1^7 + \mathcal{O}(\phi^{-2})\,.$$

[19] Here μ labels the two roots of the last summand in (3.51) and ζ corresponds to a choice of the sign in (3.42).

Taking the root $x = 0$ imposes a constraint on the x_i, and it allows us to rewrite the terms in the first two brackets as

$$(y_0 z_1^{-1/3} + y_1) + (y_2 + y_3), \qquad y_0^3 = y_1 y_2 y_3 \ . \tag{3.53}$$

This is the equation for the Riemann surface Σ representing the mirror of $\mathcal{O}(-3)_{\mathbb{P}^2}$ [51, 52]. It can be verified that the factors in the holomorphic (3,0) form work out as well. After a final rescaling $y_0 \to z_1^{1/3} y_0$, the integral for the domain wall interpolating between the curves $C_{\alpha_1,\pm}$ becomes

$$T_1^{(+,-)}(z_1, z_2 = 0) \sim \int_{y_2=-\sqrt{z_1}}^{y_2=+\sqrt{z_1}} \ln(y_1) d \ln y_2 \ . \tag{3.54}$$

This is a 'half-cycle' on the Riemann surface, which reproduces the results for the local brane of ref. [35].

Second divisor

The same domain walls can be alternatively studied via the family of divisors

$$Q(\mathcal{D}_2) = x_5^2 + z_3 z_2^{-1/2} x_1^7 x_5 \,, \tag{3.55}$$

with the curves $C_{\alpha,\pm}$ contained in the divisor with $z_3 = -\alpha\, z_2^{1/2}$. Following the same steps as in the previous example, one recovers the superpotentials (3.46) as the integrals

$$W_a^{(\pm)} = \frac{1}{2\pi i} \int_*^{\beta_a} \frac{c}{2} B_{\tilde{l}}(u; \tfrac{1}{2}) \, \frac{d\xi}{\xi} \ , \quad a = 1, 2 \ ,$$

where $\beta_{1/2} = \pm i (z_2^{1/2} \alpha_{1/2})^{1/2}$ with $\alpha_{1/2}$ defined in eq. (3.50). The charge vector $\tilde{l} = (-7, 4, 1, 1, 1)$ describes the subsystem defined by $\mathcal{D}_2$, and $u = -z_1 z_2^3 z_3^{-7} (z_2 - z_3 + z_3^2)^4$ is the single algebraic modulus associated with it.

3.3 Degree 18 hypersurface in $\mathbb{P}_{1,1,1,6,9}$

The degree 18 manifold is one of the first examples, for which Ooguri–Vafa invariants for supersymmetric branes with a large volume phase have been obtained from open-string mirror symmetry [16]. Here we study branes near critical points of the generic type. The charge vectors of the GLSM for the A model manifold are given by:

	0	1	2	3	4	5	6
l^1	-6	0	0	0	2	3	1
l^2	0	1	1	1	0	0	-3

.

In homogeneous coordinates of $\mathbb{P}_{1,1,1,6,9}$ the hypersurface constraint for the mirror manifold becomes

$$P = x_1^{18} + x_2^{18} + x_3^{18} + x_4^3 + x_5^2 + \psi\,(x_1 x_2 x_3 x_4 x_5) + \phi\,(x_1 x_2 x_3)^6 \ , \tag{3.56}$$

where $\psi = z_1^{-1/6} z_2^{-1/18}$ and $\phi = z_2^{-1/3}$. The Greene-Plesser orbifold group acts as $x_i \to \lambda_k^{g_{k,i}} x_i$ with $\lambda_1^{18} = 1$, $\lambda_2^6 = 1$ and the weights

$$\mathbb{Z}_{18}: \; g_1 = (1, -1, 0, 0, 0), \quad \mathbb{Z}_6: \; g_2 = (0, 1, 3, 2, 0) \,. \tag{3.57}$$

In this geometry we consider the curves

$$\begin{aligned} C_{\alpha,\pm} = \{x_2 = \eta_1 x_1, \; x_5 = \eta_2 x_3^9 - \frac{\psi}{2} x_1 x_2 x_3 x_4, \; x_4 = \psi^2 \alpha (x_1 x_2 x_3)^2\} \,, \\ \eta_1^{18} = \eta_2^2 = -1, \quad \alpha^3 - \frac{1}{4}\alpha^2 + \frac{\phi}{\psi^6} = 0 \,, \end{aligned} \tag{3.58}$$

where different choices for $\eta_{1,2}$ are identified under the orbifold group as $(\eta_1, \eta_2, \alpha) \sim (\eta_1 \lambda_1^2 \lambda_2^{-1}, \eta_2 \lambda_2^3, \alpha)$, and we distinguish the curves $C_{\alpha,+}$ and $C_{\alpha,-}$ by the orbits of the labels (η_1, η_2, α) under this orbifold action. Specifically the orbits $C_{\alpha,\pm}$ contain the components $\eta_1^9 = \pm i$ for fixed $\eta_2 = i$ and fixed α, respectively.

Divisor geometry and tensions

We study the family of divisors

$$Q(\mathcal{D}) = x_2^{18} + z_3 x_1^{18} \,. \tag{3.59}$$

The periods on this family are captured by the GLSM with charges

$$\begin{array}{c|c|ccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \tilde{l}^1 & -6 & 0 & 0 & 2 & 3 & 1 \\ \tilde{l}^2 & 0 & 1 & 2 & 0 & 0 & -3 \end{array} \,,$$

where the two algebraic moduli are $u_1 = z_1$ and $u_2 = -\frac{z_2}{z_3}(1 - z_3)^2$. The exceptional solutions

$$\begin{aligned} \pi_1(u) &= \frac{c_1}{2} B_{\{\tilde{l}^1, \tilde{l}^2\}}(u_1, u_2; 0, \tfrac{1}{2}) = -\frac{c_1}{2\pi} \sqrt{u_2} \; {}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}, -\tfrac{1}{2}, 432 u_1\right) + \mathcal{O}(u_2^{3/2}) \,, \\ \pi_2(u) &= \frac{c_2}{2} B_{\{\tilde{l}^1, \tilde{l}^2\}}(u_1, u_2; \tfrac{1}{2}, \tfrac{1}{2}) = \frac{2048 c_2}{\pi} u_1^{3/2} \sqrt{u_2} \; {}_2F_1\left(\tfrac{5}{3}, \tfrac{7}{3}, \tfrac{5}{2}, 432 u_1\right) + \mathcal{O}(u_2^{3/2}) \,, \end{aligned} \tag{3.60}$$

vanish at the critical point $u_2 = 0$. Similarly to eq. (3.14) we define off-shell superpotentials by

$$\mathcal{W}_a^{(\pm)}(z_1, z_2, z_3) = \frac{1}{2\pi i} \int_{\xi_0}^{\pm\sqrt{z_3}} \pi_a(u(z_1, z_2, \xi^2)) \, \frac{d\xi}{\xi} \,,$$

with the fixed reference point ξ_0. For $\xi_0 = i$ the contribution of the reference point vanishes, and at the critical value $z_3 = 1$ we arrive at the on-shell superpotentials $W_a^{(\pm)}$, where the $\pm$-label is now correlated with the orbits of the curves (3.58)

$$\begin{aligned} W_1^{(\pm)} &= \pm \frac{c_1}{8} \sum_{n_i \geq 0} \frac{\Gamma(6n_1 + 1) \; z_1^{n_1} z_2^{n_2 + \frac{1}{2}}}{\Gamma(2n_1 + 1) \Gamma(3n_1 + 1) \Gamma\left(n_1 - 3n_2 - \frac{1}{2}\right) \Gamma\left(n_2 + \frac{3}{2}\right)^3} \,, \\ W_2^{(\pm)} &= \pm \frac{c_2}{8} \sum_{n_i \geq 0} \frac{\Gamma(6n_1 + 4) \; z_1^{n_1 + \frac{1}{2}} z_2^{n_2 + \frac{1}{2}}}{\Gamma(2n_1 + 2) \Gamma\left(3n_1 + \frac{5}{2}\right) \Gamma(n_1 - 3n_2) \Gamma\left(n_2 + \frac{3}{2}\right)^3} \,. \end{aligned} \tag{3.61}$$

They can be expressed in terms of the bulk generating function as

$$W_1^{(\pm)} = \pm\frac{c_1}{8}\,B_{\{l^1,l^2\}}\left(z_1,z_2;0,\tfrac{1}{2}\right)\ ,\quad W_2^{(\pm)} = \pm\frac{c_2}{8}\,B_{\{l^1,l^2\}}\left(z_1,z_2;\tfrac{1}{2},\tfrac{1}{2}\right)\ . \tag{3.62}$$

Again these functions can be also obtained as solutions to the large GKZ system (2.21) of the relative cohomology problem. For the family (3.59) we add the additional charge vector

	0	1	2	3	4	5	6	7	8
l^3	0	-1	1	0	0	0	0	1	-1

.

This leads to the generalized hypergeometric system

$$\begin{aligned}
\tilde{\mathcal{L}}_1 &= \theta_1(\theta_1-3\theta_2)-12z_1(6\theta_1+1)(6\theta_1+5)\,,\\
\tilde{\mathcal{L}}_2 &= \theta_2(\theta_2-\theta_3)(\theta_2+\theta_3)-z_2(\theta_1-3\theta_2)(\theta_1-3\theta_2-1)(\theta_1-3\theta_2-2)\,,\\
\tilde{\mathcal{L}}_3 &= \theta_3(\theta_2+\theta_3)+z_3\theta_3(\theta_2-\theta_3)\,,
\end{aligned} \tag{3.63}$$

annihilating the relative period integrals. There are two solutions with a minimum at the critical locus $\ln(z_3)=0$ that restrict to the on-shell superpotentials $W_1^{(\pm)}$ and $W_2^{(\pm)}$, respectively.

To characterize the on-shell superpotentials $W_a^{(\pm)}$ as solutions to an inhomogeneous Picard-Fuchs equation we note that

$$\tilde{\mathcal{L}}_1 = \mathcal{L}_1^{bulk},\qquad \tilde{\mathcal{L}}_2 = \mathcal{L}_2^{bulk}-\theta_2\theta_3^2\,.$$

So only the second operator acquires an inhomogeneous term, which is determined by the leading part of the surface periods $\pi_a(u)$. Acting with $\theta_2\theta_3$ on the terms in (3.60) one obtains the inhomogeneous Picard-Fuchs equations

$$\begin{aligned}
A_1^{(\pm)} &:= \ \mathcal{L}_2^{bulk}W_1^{(\pm)} \ = \pm\frac{-c_1}{16\pi^2}\sqrt{z_2}\ {}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};-\tfrac{1}{2};432z_1\right)\ ,\\
A_2^{(\pm)} &:= \ \mathcal{L}_2^{bulk}W_2^{(\pm)} \ = \pm\frac{4096c_2}{16\pi^2}z_1^{3/2}\sqrt{z_2}\ {}_2F_1\left(\tfrac{5}{3},\tfrac{7}{3};\tfrac{5}{2};432z_1\right)\ .
\end{aligned} \tag{3.64}$$

To find the geometric domain wall tensions, we note that the three roots α_ℓ of the cubic equation (3.58) can be written as

$$\alpha_\ell \ = \ \frac{1}{12}\left(1+e^{\frac{2\pi i}{3}(\ell-1)}\,\Delta+e^{-\frac{2\pi i}{3}(\ell-1)}\,\frac{1}{\Delta}\right)\ ,\quad \ell=1,2,3\ ,$$

with

$$\Delta \ = \ \sqrt[3]{1-864z_1+2\sqrt{432z_1\,(432z_1-1)}}\ .$$

Under a monodromy $z_1^{-1}\to e^{2\pi i}\,z_1^{-1}$ around $z_1=\infty$, Δ transforms as $\Delta\to e^{\frac{2\pi i}{3}}\Delta$ and the three roots are permuted according to $\alpha_\ell\to\alpha_{\ell+1}$. On the curves $C_{\alpha,\pm}$ the monodromy acts as the $\mathbb{Z}_6$ symmetry

$$M(z_1=\infty):\quad \begin{pmatrix}\alpha_{1,\pm}\\ \alpha_{2,\pm}\\ \alpha_{3,\pm}\end{pmatrix}\mapsto\begin{pmatrix}\alpha_{2,\mp}\\ \alpha_{3,\pm}\\ \alpha_{1,\pm}\end{pmatrix}\ .$$

It follows that the domain walls between the curves $C_{\alpha,\pm}$ for fixed α must be permuted under the monodromy as well. To this end note that the hypergeometric functions in eq. (3.64) are solutions of the same hypergeometric differential equation and in fact are related by monodromy. Indeed, for $c \equiv c_1 \equiv c_2$, the inhomogeneous pieces can be expressed in terms of the single rational expression

$$f_1(\alpha) = 0 \ , \qquad f_2(\alpha) = \frac{c}{4\pi^2}\frac{1-12\alpha}{\psi^9\,\alpha^3\,(1-6\alpha)^{\,3}} \ , \tag{3.65}$$

as

$$f_2(\alpha_1) = -2A_2^{(+)} \, , \qquad f_2(\alpha_2) = -A_1^{(+)} + A_2^{(+)} \, , \qquad f_2(\alpha_3) = A_1^{(+)} + A_2^{(+)} \, . \tag{3.66}$$

From the above we obtain the following linear combinations for the geometric superpotentials

$$W_{\alpha_1}^{(\pm)} = 2\,W_2^{(\pm)} \, , \qquad W_{\alpha_2}^{(\pm)} = W_1^{(\pm)} - W_2^{(\pm)} \, , \qquad W_{\alpha_3}^{(\pm)} = W_1^{(\pm)} + W_2^{(\pm)} \, , \tag{3.67}$$

which satisfy $\mathcal{L}_2^{bulk}\,W_\alpha^{(\pm)} = \pm f(\alpha)$.

The inhomogeneous term (3.66) becomes singular at the zeros of the open-string discriminant

$$\Delta_\alpha = z_1(1-432z_1) \, ,$$

of the cubic equation, where two roots coincide. This leads to the appearance of tensionless domain walls

$$\begin{array}{lllll} z_1 = 0 & \Rightarrow & (\alpha_\ell) = (\frac{1}{4},\,0,\,0) & \Rightarrow & \begin{array}{l} T_{\alpha_1,\alpha_1}^{(+,-)} = W_{\alpha_1}^{(+)} - W_{\alpha_1}^{(-)} = 0 \\ T_{\alpha_2,\alpha_3}^{(\pm,\pm)} = W_{\alpha_2}^{(\pm)} - W_{\alpha_3}^{(\pm)} = 0 \end{array} \ , \\ z_1 = \frac{1}{432} & \Rightarrow & (\alpha_\ell) = (\frac{1}{6},-\frac{1}{12},\frac{1}{6}) & \Rightarrow & \begin{array}{l} T_{\alpha_2,\alpha_2}^{(+,-)} = W_{\alpha_2}^{(+)} - W_{\alpha_2}^{(-)} = 0 \\ T_{\alpha_1,\alpha_3}^{(\pm,\pm)} = W_{\alpha_1}^{(\pm)} - W_{\alpha_3}^{(\pm)} = 0 \end{array} \ . \end{array} \tag{3.68}$$

A-model expansion

In the Tab. 3 we list the integer invariants of the superpotentials $W_a^{(+)}$ obtained with the modified multicover formulas (3.24) and (3.25) for the normalization $c = c_1 = c_2 = 1$.

In the limit $q_1 \to 0$ the superpotential $W_1^{(+)}$ reproduces the numbers $n_k^{[3]}$ of the local Calabi-Yau geometry $\mathcal{O}(-3)_{\mathbb{P}^2}$ given in Tab. 10 in sect. A.2. Therefore in this local limit the domain wall between the curves $C_{\alpha_2,+}$ and $C_{\alpha_3,-}$, which yields the on-shell tension $T_{\alpha_2,\alpha_3}^{(+-)} \equiv W_{\alpha_2}^{(+)} - W_{\alpha_3}^{(-)} = 2\,W_1^{(+)}$, becomes equivalent to the local domain wall of the local threefold $\mathcal{O}(-3)_{\mathbb{P}^2}$ for the numbers $n_k^{[3]}$. The on-shell superpotentials $W_2^{(\pm)}$ vanish in this limit and give rise to tensionless domain walls (3.68).

$\frac{1}{2} \cdot n^{(1,+)}_{d_1,d_2}$

$q_1 \backslash q_2^{1/2}$	1	3	5	7	9
0	1	-1	5	-42	429
1	-270	270	-2430	27270	-351000
2	-35235	0	467775	-7767495	131193270
3	-1129110	-3171960	-56432160	1346568000	-30388239450
4	-19625112	-9840669480	18001000575	-268964593065	6132575901195
5	-237548052	-4228413761754	2588348258640	38534260978296	-1115308309663386
6	-2241975315	-593578595396565	241002579933810	-5655664165568310	165340822601302875

$\frac{1}{8192} \cdot n^{(2,+)}_{d_1,d_2}$

$q_1^{1/2} \backslash q_2^{1/2}$	1	3	5	7	9
1	0	0	0	0	0
3	-1	0	0	0	0
5	-54	108	-270	1728	-15444
7	-1215	-24300	99630	-918540	10783125
9	-17290	-60310547	-15819570	220135880	-3485260710

Table 3: Disc invariants for the on-shell superpotentials $W_a^{(+)}$ of the threefold $\mathbb{P}_{1,1,1,6,9}[18]$.

Non-compact limit

We exhibit the non-compact limit by redefining the projective coordinate of $\mathbb{P}_{1,1,1,6,9}/(\mathbb{Z}_{18}\times\mathbb{Z}_6)$ according to

$$y_1 \,=\, x_1^6 \ , \quad y_2 \,=\, x_2^6 \ , \quad y_3 \,=\, x_3^6 \ , \quad x \,=\, x_5 \ , \quad z \,=\, x_5 + \psi\, x_1 x_2 x_3 x_4 \ ,$$

where $y_\ell \in \mathbb{C}^*$, $x, z \in \mathbb{C}$. In these local coordinates the Greene-Plesser orbifold group reduces to $\mathbb{Z}_3$. It acts on the coordinates y_ℓ as $y_\ell \to \lambda^\ell y_\ell$, with $\lambda^3 = 1$, while the coordinates x, z remain invariant. In the limit $z_1 \to 0$, which is mirror symmetric to the limit $q_1 \to 0$, we arrive at the local Calabi-Yau geometry

$$0 \,=\, y_1^3 + y_2^3 + y_3^3 + z_2^{1/3}\, y_1 y_2 y_3 + x\, z + \mathcal{O}(\sqrt{z_1}) \ ,$$

together with the associated local holomorphic three-form Ω. This limit has already been studied in detail in ref. [16]. The local geometry is related to the (mirror) cubic elliptic curve with the points $y_\ell = 0$ removed, and it captures the local mirror of the non-compact threefolds $\mathcal{O}(-3)_{\mathbb{P}^2}$ studied in app. A.2. This explains the appearance of the disc invariants $n_k^{[3]} = n_{0,d_2}^{(1,+)}$ in Tab. 3.

3.4 Degree 9 hypersurface in $\mathbb{P}_{1,1,1,3,3}$

Ooguri-Vafa invariants for supersymmetric branes with a large volume phase on this manifold have been computed in ref. [16]. Here we study branes near critical points of the generic type. The charge vectors of the GLSM for the A model manifold are given by:

$$\begin{array}{c|c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline l^1 & -3 & 0 & 0 & 0 & 1 & 1 & 1 \\ l^2 & 0 & 1 & 1 & 1 & 0 & 0 & -3 \end{array} \ .$$

The hypersurface constraint for the mirror manifold, written in homogeneous coordinates of $\mathbb{P}_{1,1,1,3,3}$, is

$$P = x_1^9 + x_2^9 + x_3^9 + x_4^3 + x_5^3 - \psi\,(x_1x_2x_3x_4x_5) + \phi\,(x_1x_2x_3)^3\,, \tag{3.69}$$

where $\psi = z_1^{-1/3} z_2^{-1/9}$ and $\phi = z_2^{-1/3}$. The Greene-Plesser orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with $\lambda_1^9 = \lambda_2^9 = 1$, $\lambda_3^3 = 1$ and weights

$$\mathbb{Z}_9 :\, g_1 = (1,-1,0,0,0), \quad \mathbb{Z}_9 :\, g_2 = (1,0,-1,0,0), \quad \mathbb{Z}_3 :\, g_2 = (0,0,0,1,-1)\,. \tag{3.70}$$

In this geometry, we study the family of divisors

$$Q(\mathcal{D}) = x_2^9 + z_3 x_1^9\,, \tag{3.71}$$

near the point $z_3 = -1$. The Calabi-Yau threefold is an elliptic fibration over $\mathbb{P}^2$ similar to the previous example and the steps of the computation of the periods of the relative cohomology group defined by the divisor $\mathcal{D}$ are straightforward. Despite these similarities, we could not identify a complete intersection representation of the type (3.58) for an appropriate curve. In the following we proceed to compute the superpotential and the disc invariants for the critical point without knowing such an explicit representation.

The surface periods defined by the family (3.71) are captured by the GLSM with charges

$$\begin{array}{c|c|ccccc} & 0 & 1 & 3 & 4 & 5 & 6 \\ \hline \tilde{l}^1 & -3 & 0 & 0 & 1 & 1 & 1 \\ \tilde{l}^2 & 0 & 2 & 1 & 0 & 0 & -3 \end{array}\,,$$

depending on the two algebraic moduli $u_1 = z_1$ and $u_2 = -\frac{z_2}{z_3}(1-z_3)^2$. The exceptional solutions

$$\begin{aligned} \pi_1 &= \frac{c_1}{2}\; B_{\{\tilde{l}^1,\tilde{l}^2\}}(u_1,u_2;0,\tfrac{1}{2}) = \frac{-c_1}{2\pi}\,\sqrt{u_2}\;{}_2F_1(\tfrac{1}{3},\tfrac{2}{3},-\tfrac{1}{2},27z_1) + \mathcal{O}(u_2^{3/2})\,, \\ \pi_2 &= \frac{c_2}{2}\; B_{\{\tilde{l}^1,\tilde{l}^2\}}(u_1,u_2;\tfrac{1}{2},\tfrac{1}{2}) = \frac{105c_2}{2\pi}\,\sqrt{u_2}\;z_1^{3/2}\;{}_2F_1(\tfrac{11}{6},\tfrac{13}{6},\tfrac{5}{2},27z_1) + \mathcal{O}(u_2^{3/2})\,, \end{aligned} \tag{3.72}$$

vanish at the point $z_3 - 1 = 0 = u_2$. The sign of the root $\sqrt{z_3}$ distinguishes the two sheets of the coordinate change $x_1^2 = \tilde{x}_1$ similarly as in eq. (3.14). Integrating along similar contours as in that case, we obtain the superpotentials

$$\begin{aligned} W_1^{(\pm)} &= \pm\frac{c_1}{8}\sum_{n_i\geq 0}\frac{\Gamma\left(3n_1+1\right)\,z_1^{n_1} z_2^{n_2+\frac{1}{2}}}{\Gamma\left(n_1+1\right)^2\Gamma\left(n_1-3n_2-\frac{1}{2}\right)\Gamma\left(n_2+\frac{3}{2}\right)^3}\,, \\ W_2^{(\pm)} &= \pm\frac{c_2}{8}\sum_{n_i\geq 0}\frac{\Gamma\left(3n_1+\frac{5}{2}\right)\,z_1^{n_1+\frac{1}{2}} z_2^{n_2+\frac{1}{2}}}{\Gamma\left(n_1+\frac{3}{2}\right)^2\Gamma\left(n_1-3n_2\right)\Gamma\left(n_2+\frac{3}{2}\right)^3}\,, \end{aligned} \tag{3.73}$$

or equivalently, expressed in terms of the bulk generating function

$$W_1^{(\pm)} = \pm\frac{c_1}{8}\;B_{\{l^1,l^2\}}\left(z_1,z_2;0,\tfrac{1}{2}\right)\,, \qquad W_2^{(\pm)} = \pm\frac{c_2}{8}\;B_{\{l^1,l^2\}}\left(z_1,z_2;\tfrac{1}{2},\tfrac{1}{2}\right)\,. \tag{3.74}$$

These functions are solutions to the large GKZ system (2.21) of the relative cohomology problem. For the family (3.71) the additional extended charge vector is

$$\begin{array}{cc|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline l^3 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{array},$$

which, together with the charge vectors of the threefold, gives rise to the differential operators according to eq. (2.21)

$$\begin{aligned} \tilde{\mathcal{L}}_1 &= \theta_1(\theta_1 - 3\theta_2) - 3z_1(3\theta_1 + 1)(3\theta_1 + 2) , \\ \tilde{\mathcal{L}}_2 &= (\theta_2 - \theta_3)(\theta_2 + \theta_3)\theta_2 - z_2(\theta_1 - 3\theta_2)(\theta_1 - 3\theta_2 - 1)(\theta_1 - 3\theta_2 - 2) , \\ \tilde{\mathcal{L}}_3 &= \theta_3(\theta_2 + \theta_3) + z_3\,\theta_3(\theta_2 - \theta_3) . \end{aligned} \tag{3.75}$$

The solutions to these operators are the relative period integrals. In particular there are two solutions with a minimum at the critical locus $\ln(z_3) = 0$, which restrict to the on-shell superpotentials $W_1^{(\pm)}$ and $W_2^{(\pm)}$, respectively.

To characterize the critical superpotentials $W_a^{(\pm)}$ as solutions to an inhomogeneous Picard-Fuchs equation, we observe

$$\tilde{\mathcal{L}}_1 = \mathcal{L}_1^{bulk}, \qquad \tilde{\mathcal{L}}_2 = \mathcal{L}_2^{bulk} - \theta_2\theta_3^2 .$$

So only the second operator acquires an inhomogeneous term, which is determined by the leading part of the surface periods π_a. Acting with $\theta_2\theta_3$ on the leading coefficients of (3.72) one obtains the inhomogeneous Picard-Fuchs equations

$$\begin{aligned} \mathcal{L}_2^{bulk} W_1^{(\pm)} &= \mp \frac{c_1}{16\pi^2}\sqrt{z_2}\, {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; -\tfrac{1}{2}; 27z_1\right) , \\ \mathcal{L}_2^{bulk} W_2^{(\pm)} &= \pm \frac{105c_2}{16\pi^2} z_1^{3/2}\sqrt{z_2}\, {}_2F_1\left(\tfrac{11}{6}, \tfrac{13}{6}; \tfrac{5}{2}; 27z_1\right) . \end{aligned} \tag{3.76}$$

The inhomogeneous terms are again solutions to the same hypergeometric equation and related by monodromy. The differential operator is obtained by specializing the Picard-Fuchs operator of the surface $\mathcal{L}_1^{\mathcal{D}} = \tilde{\theta}_1(\tilde{\theta}_1 - 3\tilde{\theta}_2) - 3u_1(3\tilde{\theta}_1 + 1)(3\tilde{\theta}_1 + 2)$ to the critical point $u_2 = 0$:

$$\mathcal{L}^{inh}\, f_2(z_1, z_2) = 0\,, \qquad \mathcal{L}^{inh} = \mathcal{L}_1^{\mathcal{D}}|_{\hat{z}_{crit}} = (1-z)z\frac{d^2}{dz^2} + \left(-\frac{1}{2} - 2z\right)\frac{d}{dz} - \frac{2}{9}\,,$$

with $\tilde{\theta}_a = u_a\frac{d}{du_a}$, $z = 27z_1$. The specialization to the leading term in the limit $u_2 = 0$ is achieved by setting $\theta_2 = \frac{1}{2}$. Similarly as in the other examples one can verify that the hypergeometric functions (3.76) can be written in closed form.

In Tab. 4 we list the integer invariants obtained with the modified multicover formula (3.24), (3.25) for the normalization $c_1 = c_2 = 1$. Similarly as in the previous examples, the hypersurface degenerates to the non-compact threefold (3.53) in the limit $z_1 \to 0$ [16]. This explains the appearance of the invariants $n^{[3]}$ (c.f. Tab. 10) in the superpotential $W_1^{(+)}$, which are listed in the first row of the first table in Tab. 4.

$\frac{1}{2} \cdot n^{(1,+)}_{d_1,d_2}$

$q_1 \setminus q_2^{1/2}$	1	3	5	7	9	11
0	1	-1	5	-42	429	-4939
1	-27	27	-243	2727	-35100	487647
2	-243	0	4131	-71442	1230795	-21333942
3	-1347	-2295	-33804	979800	-24220836	544584789
4	-6021	-231876	532575	-10061955	319551804	-9298367514
5	-22356	-7276878	5101407	73610289	-3196953927	117194205483

$\frac{1}{2} \cdot n^{(2,+)}_{d_1,d_2}$

$q_1^{1/2} \setminus q_2^{1/2}$	1	3	5	7	9
1	0	0	0	0	0
3	-105	0	0	0	0
5	-567	1134	-2835	18144	-162162
7	-2916	-18954	81648	-826686	10133100
9	-11904	-1421850	-498555	13289664	-255008817

Table 4: Disc invariants for the on-shell superpotentials $W_a^{(+)}$ of the threefold $\mathbb{P}_{1,1,1,3,3}[9]$.

3.5 Degree 12 hypersurface in $\mathbb{P}_{1,1,2,2,6}$

Ooguri–Vafa invariants for supersymmetric branes with a large volume phase on this manifold have been computed in ref. [30]. Here we study branes near critical points of the generic type. The critical value of the superpotential for these branes was computed already in ref. [35]. This gives a check on the off-shell superpotential obtained from the GKZ system for the relative periods by restriction to the critical point. The charges of the GLSM for the A-model manifold are given by:

$$\begin{array}{c|c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline l^1 & \text{-6} & 0 & 0 & 1 & 1 & 3 & 1 \\ l^2 & 0 & 1 & 1 & 0 & 0 & 0 & \text{-2} \end{array} .$$

The hypersurface constraint for the mirror manifold in homogeneous coordinates of $\mathbb{P}_{1,1,2,2,6}$ is

$$P = x_1^{12} + x_2^{12} + x_3^6 + x_4^6 + x_5^2 + \psi\, x_1 x_2 x_3 x_4 x_5 + \phi\,(x_1 x_2)^6\,, \tag{3.77}$$

where $\psi = z_1^{-1/6} z_2^{-1/12}$ and $\phi = z_2^{-1/2}$. The Greene-Plesser orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with generators $\lambda_1^6 = \lambda_2^6 = \lambda_3^2 = 1$ and the weights

$$\mathbb{Z}_6 :\, g_1 = (1,0,-1,0,0)\,, \quad \mathbb{Z}_6 :\, g_2 = (1,0,0,-1,0)\,, \quad \mathbb{Z}_2 :\, g_3 = (1,0,0,0,-1)\,. \tag{3.78}$$

In this geometry we consider the same curves as in ref. [35],

$$\begin{aligned} C_{\alpha,\eta} = \{x_3 = \eta_1 x_1^2\,,\ x_4 = \eta_2 x_2^2\,,\ x_5 = \alpha x_1 x_2 x_3 x_4\}\,,& \\ \eta_1^6 = \eta_2^6 = -1\,, \qquad \alpha^2 + \psi\alpha + \frac{\phi}{(\eta_1\eta_2)^2} = 0\,,& \end{aligned} \tag{3.79}$$

which under the orbifold group are identified as $(\eta_1, \eta_2, \alpha) \sim (\eta_1 \lambda_1^3 \lambda_2^2, \eta_2 \lambda_2, \alpha)$. The 36 choices for η_1 and η_2 form 3 orbits of length 12. Together with the two choices for α there are 6 different curves, that we choose to label by (α, η), where $\eta = (\eta_1\eta_2)^{-2}$ and $\eta^3 = 1$.

Divisor geometry and tensions

We study the family of divisors

$$Q(\mathcal{D}) = x_5^2 - z_3 z_1^{-1/6} z_2^{-1/12} x_1 x_2 x_3 x_4 x_5 \,, \tag{3.80}$$

which contains the curves $C_{\alpha,\eta}$ at the critical points $z_3 = \alpha z_1^{1/6} z_2^{1/12}$. Note that the chosen open coordinate z_3 arises naturally in the associated non-compact fourfold defined by the additional charge vector

	0	1	2	3	4	5	6	7	8
l^3	-1	0	0	0	0	1	0	-1	1

.

Derivatives of the relative periods with respect to z_3 are related to the surface periods of the intersection $Q(\mathcal{D}) = 0 = P$. The relevant surface is captured by the GLSM with the charges

	0	1	2	3	4	5
$\tilde{l}^1$	-3	0	0	1	1	1
$\tilde{l}^2$	0	1	1	0	0	-2

.

The algebraic moduli of the surface are related to those of the threefold by $u_1 = -\frac{z_1}{z_3^3(1+z_3)^3}$ and $u_2 = z_2$.

In the moduli of the intersection surface the critical points $z_3 = \alpha z_1^{1/6} z_2^{1/12} \equiv \tilde{\alpha}$ are given in terms of the condition $u_1 = u_2$ or equivalently in terms of $-\frac{z_1}{z_2} = z_3^3(1+z_3)^3$. Then the characteristic equation for the curves $C_{\alpha,\eta}$ becomes

$$\eta y + \tilde{\alpha}(1+\tilde{\alpha}) = 0 \,, \qquad y \equiv \left(\frac{z_1}{z_2}\right)^{1/3} \,, \tag{3.81}$$

with $\eta^3 = 1$, and the critical points are given by

$$\tilde{\alpha}_\pm(\eta) = z_1^{1/6} z_2^{1/12} \alpha_\pm(\eta) = \frac{1}{2}\left(-1 \pm \sqrt{1-4\eta y}\right) \,. \tag{3.82}$$

Hence the critical points $\tilde{\alpha}(\eta)$ are in one-to-one correspondence to the labels (α,η) of the curves $C_{\alpha,\eta}$.

The solutions of this subsystem can be generated with the Frobenius method from the generating function,

$$B_{\{\tilde{l}^1,\tilde{l}^2\}}(u_1,u_2;\rho_1,\rho_2) = \sum_{n_i \in \mathbb{Z}+\rho_i} \frac{\Gamma(1+3n_1)\, u_1^{n_1} u_2^{n_2}}{\Gamma(1+n_1)^2\, \Gamma(1+n_1-2n_2)\, \Gamma(1+n_2)^2} \,. \tag{3.83}$$

The linear combination

$$\pi(u_1,u_2) = \frac{c}{2\pi i}\, \left(\partial_{\rho_1} B_{\{\tilde{l}^1,\tilde{l}^2\}} - \partial_{\rho_2} B_{\{\tilde{l}^1,\tilde{l}^2\}}\right)\big|_{\rho_i=0} := c(t_1 - t_2) \,, \tag{3.84}$$

vanishes at $u_1 = u_2$ or equivalently at the critical points $z_3 = \tilde{\alpha}$ of eq. (3.81). Note that t_1 and t_2 are the volumes of two generators of $H_2(K3,\mathbb{Z})$, and the zero of the period arises from the coincidence of their volumes.

In order to derive the superpotentials we need to integrate the surface periods $\pi(u)$. Note that for the divisor family (3.80) the induced holomorphic two form of the embedding surface differs from the canonically normalized holomorphic two form of the corresponding isogenic K3 surface by a moduli dependent pre-factor. As a consequence the relation (2.14) must also be modified by a moduli-dependent measure factor [30]

$$2\pi i\,\theta_{z_3}\mathcal{W}(z_1,z_2,z_3)=\frac{1}{1+z_3}\pi(u_1,u_2)\ ,$$

where now both the superpotential $\mathcal{W}$ and the surface period $\pi(u_1,u_2)$ are canonically normalized. Thus integrating the surface period $\pi(u_1,u_2)$ together with the measure factor according to

$$W^{(\alpha_\pm,\eta)}=\frac{1}{2\pi i}\int_*^{\tilde{\alpha}_\pm(\eta)}\frac{1}{1+z_3}\,\pi(z_3)\frac{dz_3}{z_3}\,, \tag{3.85}$$

we find the on-shell superpotentials for the curves $C_{(\alpha,\eta)}$

$$W^{(\alpha_\pm,\eta)}(y,z_2)=\mp\,\frac{c}{4\pi^2}\left(\frac{3}{2}S_0\,\log(-\eta y)^2+(S_1-S_2)\,\log(-\eta y)+\frac{5\pi^2}{2}+S_{\alpha_+,\eta}(y,z_2)\right)\,. \tag{3.86}$$

Here S_0,S_1 and S_2 are the power-series[20]

$$\begin{aligned}
S_0&=\sum_{n_i\geq 0}\frac{\Gamma(1+6n_1)}{\Gamma(1+3n_1)\Gamma(1+n_1)^2\,\Gamma(1+n_1-2n_2)\Gamma(1+n_2)^2}\,y^{3n_1}\,z_2^{n_1+n_2}\\
&=1+120y^3z_2+83160y^6z_2^2+166320y^6z_2^3+81681600y^9z_2^3+\dots\ ,\\
S_1&=\sum_{n_i\geq 0}\frac{\Gamma(1+6n_1)\,(6\psi(1+n_1)-3\psi(1+3n_1)-2\psi(1+n_1)-\psi(1+n_1-2n_2))}{\Gamma(1+3n_1)\Gamma(1+n_1)^2\,\Gamma(1+n_1-2n_2)\Gamma(1+n_2)^2}\,y^{3n_1}\,z_2^{n_1+n_2}\\
&=-z_2+744y^3z_2+-\frac{3z_2^2}{2}+120y^3z_2^2+562932y^6z_2^2+\dots\ ,\\
S_2&=\sum_{n_i\geq 0}\frac{2\Gamma(1+6n_1)\,(\psi(1+n_1-2n_2)-\psi(1+n_2))}{\Gamma(1+3n_1)\Gamma(1+n_1)^2\,\Gamma(1+n_1-2n_2)\Gamma(1+n_2)^2}\,y^{3n_1}\,z_2^{n_1+n_2}\\
&=2z_2+240y^3z_2+3z_2^2-240y^3z_2^2+249480y^6z_2^2+\dots\ ,
\end{aligned}$$

while the instanton part reads

$$S_{\alpha_+,\eta}=(6(\eta y)+z_2)+\left(\frac{9(\eta y)^2}{2}+\frac{15z_2^2}{4}\right)+\left(\frac{20y^3}{3}+81(\eta y)^2z_2+\frac{191z_2^3}{18}\right)+\dots\,. \tag{3.87}$$

In ref. [35] the on-shell superpotentials (3.86) were obtained as the solutions to inhomogeneous Picard-Fuchs equations. To calculate these inhomogeneous terms, we rewrite the bulk operators

$$\begin{aligned}
\mathcal{L}_1^{bulk}&=\theta_1^2(\theta_1-2\theta_2)-8z_1(6\theta_1+1)(6\theta_1+3)(6\theta_1+5)\ ,\\
\mathcal{L}_2^{bulk}&=\theta_2^2-z_2(\theta_1-2\theta_2)(\theta_1-2\theta_2-1)\ ,
\end{aligned} \tag{3.88}$$

[20] S_0 is the fundamental closed string period and S_a, $a=1,2$, the series part of the single logarithmic closed string periods $(2\pi i)t_a=\log(z_a)+S_a$, which determine the closed string mirror map. However there is no double logarithimic closed string period that has the same classical terms as eq. (3.86).

$q_y \backslash q_{z_2}$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	6	6	0	0	0	0	0	0	0	0
2	3	90	3	0	0	0	0	0	0	0
3	6	388	388	6	0	0	0	0	0	0
4	12	-258	2934	-258	12	0	0	0	0	0
5	30	-540	11016	11016	-540	30	0	0	0	0
6	75	-1388	67602	348774	67602	-1388	75	0	0	0
7	210	-3960	-44496	731952	731952	-44496	-3960	210	0	0
8	600	-12042	-75036	3177414	20289960	3177414	-75036	-12042	600	0
9	1800	-38236	-136672	20383740	399653208	399653208	20383740	-136672	-38236	1800

Table 5: Symmetric disc invariants for the on-shell superpotentials $W^{(\alpha_+,1)}$ of the threefold $\mathbb{P}_{1,1,2,2,6}[12]$.

in terms of the coordinates y and z_2, we act with them upon the superpotentials (3.86), and we find for the inhomogeneous terms

$$\mathcal{L}_1^{bulk}\,W^{(\alpha_\pm,\eta)} = \pm\frac{c}{4\pi^2}\left(\frac{2}{3}\eta y + 4(\eta y)^2 + 20(\eta y)^3 + \ldots\right) = \pm\frac{c}{6\pi^2}\frac{\eta y}{(1-4\eta y)^{3/2}}\ ,$$
$$\mathcal{L}_2^{bulk}\,W^{(\alpha_\pm,\eta)} = \pm\frac{c}{4\pi^2}\left(\frac{1}{3} + \frac{2}{3}\eta y + 2(\eta y)^2 + \frac{20}{3}(\eta y)^3 + \ldots\right) = \pm\frac{c}{12\pi^2}\frac{1}{(1-4\eta y)^{1/2}}\ .$$

A-model expansion

For completeness we quote in Tab. 5 the disc instantons for the on-shell superpotentials in terms of $q_y = \frac{1}{3}(z_1 - z_2) + \ldots$ and $q_{z_2} = z_2 + \ldots$ for $c = 1$. These numbers have already been computed in ref. [35] by deriving the inhomogeneous Picard-Fuchs equations. As in ref. [35] we have added a rational multiple of a closed-string period with leading behavior $\Pi = t_1 t_2 + \ldots$ to get invariants n_{d_1,d_2} symmetric under the $\mathbb{Z}_2$ symmetry $t_1 \to t_1 + t_2,\ t_2 \to -t_2;\ q_y \to q_y q_2,\ q_2 \to q_2^{-1}$. This is the Weyl symmetry of a non-perturbative $SU(2)$ gauge symmetry appearing in the type II compactification at the transition point [46, 47]. The domain wall is a singlet under this global symmetry as can be seen from the defining equation (3.79).

3.6 Degree 8 hypersurface in $\mathbb{P}_{1,1,2,2,2}$

The charge vectors of the GLSM for the A-model manifold are given by:

	0	1	2	3	4	5	6
l^1	-4	0	0	1	1	1	1
l^2	0	1	1	0	0	0	-2

The hypersurface constraint for the mirror manifold in homogeneous coordinates of $\mathbb{P}_{1,1,2,2,2}$ is

$$P = x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 + \psi\, x_1x_2x_3x_4x_5 + \phi\,(x_1x_2)^4 \tag{3.89}$$

where $\psi = z_1^{-1/4}z_2^{-1/8}$ and $\phi = z_2^{-1/2}$. The Greene-Plesser orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\,x_i$ with generators $\lambda_1^8 = 1,\ \lambda_2^4 = \lambda_3^4 = 1$ and weights

$$\mathbb{Z}_8:\ g_1 = (1,-1,0,0,0),\quad \mathbb{Z}_4:\ g_2 = (1,0,-1,0,0),\quad \mathbb{Z}_4:\ g_2 = (1,0,0,-1,0)\,. \tag{3.90}$$

In this geometry we consider the curves

$$C_\alpha = \{x_3 = \eta_1 x_1^2\,,\, x_4 = \eta_2 x_2^2\,,\, \eta_1\eta_2 x_5 = \alpha x_1 x_2\}\,,$$
$$\eta_1^4 = \eta_2^4 = -1\,, \alpha^4 + \psi\,\alpha + \phi = 0\,. \tag{3.91}$$

where $\eta_1^4 = \eta_2^4 = -1$. Under the orbifold action the curves are identified as $(\eta_1, \eta_2, \alpha) \sim (\eta_1\lambda_1^2\lambda_2^3\lambda_3^2, \eta_2\lambda_1^{-2}\lambda_3, \alpha)$. The curves are labeled by the four roots α, while under the $\mathbb{Z}_8 \times \mathbb{Z}_4^2$ orbifold action the 16 distinct choices for the phases η_1 and η_2 are identified. Thus we find four distinct orbits of curves C_α.

Divisor geometry and tensions

To compute DW tensions for these curves we study the family of divisors

$$Q(\mathcal{D}) = x_5^4 - z_3\, z_1^{-1/4} z_2^{-1/8}\, x_1x_2x_3x_4x_5\ . \tag{3.92}$$

The curves C_α are included in $\mathcal{D}$ for the critical values $z_3 = z_1^{1/4} z_2^{1/8}\,\alpha^3 \equiv \tilde\alpha$, where the new label $\tilde\alpha$ obeys the fourth order equation

$$\tilde\alpha(1+\tilde\alpha)^3 + y\, =\, 0\ , \qquad y \equiv \frac{z_1}{z_2}\ . \tag{3.93}$$

Note that the roots $\tilde\alpha$ of this fourth order equation are in one-to-one correspondence with the curves C_α.

The chosen open-string coordinate z_3 is the natural coordinate on the non-compact fourfold defined by adding

	0	1	2	3	4	5	6	7	8
l^3	-1	0	0	0	0	1	0	1	-1

to the GLSM for the A-model manifold. Periods on the intersection $Q(\mathcal{D}) = P = 0$ are captured by a GLSM with charges

	0	1	2	3	4	5
$\tilde l^1$	-3	0	0	1	1	1 ,
$\tilde l^2$	0	1	1	0	0	-2

where the algebraic moduli are $u_1 = -\frac{z_1}{z_3(1+z_3)^3}$ and $u_2 = z_2$. In these coordinates the critical points $z_3 = \tilde\alpha$ arise at $u_1 = u_2$. This condition corresponds to the fourth order equation (3.93) for the label $\tilde\alpha$.

The solutions of this subsystem can be generated with the Frobenius method from the generating function

$$B_{\{\tilde l^1,\tilde l^2\}}(u_1,u_2;\rho_1,\rho_2) = \sum_{n_i\in\mathbb{Z}+\rho_i} \frac{\Gamma(1+3n_1)\,u_1^{n_1}u_2^{n_2}}{\Gamma(1+n_1)^2\,\Gamma(1+n_1-2n_2)\,\Gamma(1+n_2)^2}\,. \tag{3.94}$$

The linear combination

$$\tau = \frac{c}{2\pi i}(\partial_{\rho_1}B_{\{\tilde l^1,\tilde l^2\}} - \partial_{\rho_2}B_{\{\tilde l^1,\tilde l^2\}})\big|_{\rho_i=0} := c(t_1-t_2)\,, \tag{3.95}$$

vanishes at the critical locus $u_1 = u_2$. Again t_1 and t_2 measure the volumes of two generators of $H_2(K3,\mathbb{Z})$ and at criticality the zero of the period arises because their volumes coincide. The four critical points $\tilde{\alpha}_k, k = 0,1,2,3$, which are given in terms of the fourth order equation (3.93), enjoy in terms of $y = \frac{z_1}{z_2}$ the expansion

$$\begin{aligned} \tilde{\alpha}_0(y) &= -y\left(1 + 3y + 15y^2 + 91y^3 + 612y^4 + 4389y^5 + 32890y^6 + \ldots\right) , \\ \tilde{\alpha}_\ell(y) &= -1 + \nu_\ell y^{1/3} + \frac{1}{3}\nu_\ell^2 y^{2/3} + \frac{\nu_\ell^3 y}{3} + \frac{35}{81}\nu_\ell^4 y^{4/3} + \frac{154}{243}\nu_\ell^5 y^{5/3} + \nu_\ell^6 y^2 + \ldots , \end{aligned} \tag{3.96}$$

with $\nu_\ell = e^{\frac{2\pi i}{3}(\ell-1)}, \ell = 1,2,3$. Similarly to the related example $\mathbb{P}_{1,1,2,2,6}$, there appears an additional measure factor for the integration of subsystem period to the superpotential, namely

$$2\pi i\,\theta_{z_3}\mathcal{W}(z_1,z_2,z_3) = \frac{1}{1+z_3}\pi(u_1,u_2) .$$

Hence, integrating the discussed subsystem period (3.95) with the additional measure factor, we obtain for the critical point $\tilde{\alpha}_0$ the on-shell superpotential

$$W^{(\alpha_0)}(y,z_2) = -\frac{c}{4\pi^2}\left(\frac{1}{2}\,S_0\,\log(-y)^2 + (S_1 - S_2)\,\log(-y) + S_{\alpha_0}(y,z_2)\right) . \tag{3.97}$$

Here S_0, S_1 and S_2 are the power-series[21]

$$\begin{aligned} S_0 &= 1 + 24yz_2 + 2520y^2z_2^2 + 5040y^2z_2^3 + 369600y^3z_2^3 + 2217600y^3z_2^4 + \ldots , \\ S_1 &= -z_2 + 104yz_2 - \frac{3}{2}z_2^2 + 24yz_2^2 + 12276y^2z_2^2 - \frac{10}{3}z_2^3 + 12yz_2^3 - \frac{35}{4}z_2^4 + \ldots , \\ S_2 &= 2z_2 + 48yz_2 + 3z_2^2 - 48yz_2^2 + 7560y^2z_2^2 + \frac{20}{3}z_2^3 - 24yz_2^3 + \frac{35}{2}z_2^4 + \ldots . \end{aligned} \tag{3.98}$$

For the instanton part S_{α_0} we get

$$S_{\alpha_0}(y,z_2) = (4y + 3z_2) + \left(7y^2 - 64yz_2 + \frac{45z_2^2}{4}\right) + \left(\frac{220y^3}{9} + 210y^2z_2 + 528yz_2^2 + \frac{191z_2^3}{6}\right) + \ldots . \tag{3.99}$$

Finally, we note that in terms of the GLSM charges suitable for the coordinates y, z_2

	0	1	2	3	4	5	6
h^1	-4	-1	-1	1	1	1	3
h^2	0	1	1	0	0	0	-2

we can express the superpotential $W^{(\alpha_0)}$ as

$$W^{(\alpha_0)}(y,z_2) \;=\; -\frac{c}{8\pi^2}\,\partial_{\rho_1}^2\,B_{\{h^1,h^2\}}(y,z_2,\rho_1,\rho_2)|_{\rho_i=0} . \tag{3.100}$$

Integrating the subsystem period with the additional measure factor to the other roots $\tilde{\alpha}_\ell(y)$ one finds similar expansions for the on-shell superpotentials $W^{(\alpha_\ell)}(y,z_2)$ associated to these roots.

[21] S_0 is the fundamental closed string period and S_a, $a = 1,2$, the series part of the single logarithmic closed string periods $(2\pi i)t_a = \log(z_a) + S_a$, which determine the closed string mirror map. However there is no double logarithimic closed-string period that has the same classical terms as eq. (3.97).

To characterize the superpotential $W^{(\alpha_0)}$ by an inhomogeneous Picard-Fuchs equation we calculate the inhomogeneous pieces with the following bulk operators

$$\mathcal{L}_1^{bulk} = \theta_1^2(\theta_1 - 2\theta_2) - 4z_1(4\theta_1+1)(4\theta_1+2)(4\theta_1+3)\,, \tag{3.101}$$
$$\mathcal{L}_2^{bulk} = \theta_2^2 - z_2(\theta_1 - 2\theta_2)(\theta_1 - 2\theta_2 - 1)\,, \tag{3.102}$$

and we obtain

$$\begin{aligned} \mathcal{L}_1^{bulk}\, W^{(\alpha_0)} &= -\frac{3\,c}{4\pi^2}\;\theta_y\, {}_3F_2\left(\tfrac{1}{4},\tfrac{2}{4},\tfrac{3}{4};\tfrac{1}{3},\tfrac{2}{3};\tfrac{256\,y}{27}\right) = 3\,\theta_y f(\alpha_0)\ , \\ \mathcal{L}_2^{bulk}\, W^{(\alpha_0)} &= -\frac{c}{4\pi^2}\;{}_3F_2\left(\tfrac{1}{4},\tfrac{2}{4},\tfrac{3}{4};\tfrac{1}{3},\tfrac{2}{3};\tfrac{256\,y}{27}\right) = f(\alpha_0)\ , \end{aligned} \tag{3.103}$$

where the label α_0 refers to the root of the quartic equation in (3.91) associated to the corresponding root $\tilde{\alpha}_0$ in eq. (3.96). As in previous examples we can also express the inhomogeneous terms as functions in the coefficients of the defining equations, i.e.

$$f(\alpha)\;=\;\frac{c}{4\pi^2}\,\frac{z_2^{1/8}\alpha}{4y^{1/4}+3z_2^{1/8}\alpha}\;=\;-\frac{c}{4\pi^2}\cdot\frac{1}{4\tilde{\alpha}+1}\ . \tag{3.104}$$

The open string discriminant is $\Delta_\alpha = y(1-\frac{256y}{27})$, with the three roots $\tilde{\alpha}_\ell$, $\ell>0$ colliding for $y=0$ at $\tilde{\alpha}_\ell=-1$, see eq. (3.96), while at $y=\frac{27}{256}$ one has $\tilde{\alpha}_0=-\frac{1}{4}=\tilde{\alpha}_1$. The inhomogeneous term (3.104) become singular at the second zero, indicating a tensionless domainwall between the curves associated with $\tilde{\alpha}_{0,1}$.

For the other on-shell superpotentials $W^{(\alpha_\ell)}(y,z_2)$, we find the same inhomogeneous terms

$$\mathcal{L}_1^{bulk}\, W^{(\alpha_\ell)}\;=\;3\,\theta_y f(\alpha_\ell)\ ,\quad \mathcal{L}_2^{bulk}\, W^{(\alpha_\ell)}\;=\;f(\alpha_\ell)\ ,\quad \ell=1,2,3\ , \tag{3.105}$$

where again the roots α_ℓ are associated to the corresponding roots $\tilde{\alpha}_\ell$.

A-model expansion

Using the standard multicover formula

$$\frac{W^{(\alpha_0)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2}\sum_k\sum_{d_{1,2}\geq 0} n^{(\alpha_1)}_{d_1,d_2}\,\frac{q_1^{kd_1}\,q_2^{kd_2}}{k^2}$$

we obtain for $c=1$ the integer invariants in Tab. 6. Here $q_y = z_1 - z_2 + \ldots$ and $q_2 = z_2 + \ldots$. Again we have added a rational multiple of a closed-string period with leading behavior $\Pi = \frac{3}{2}t_1t_2+\ldots$ to get invariants n_{d_1,d_2} symmetric under the $\mathbb{Z}_2$ symmetry $t_1\to t_1+t_2,\ t_2\to -t_2;\ q_y\to q_yq_2,\ q_2\to q_2^{-1}$. This is the Weyl symmetry of a non-perturbative $SU(2)$ gauge symmetry appearing in the type II compactification at the transition point [46, 47]. The domain wall is a singlet under this global symmetry as can be seen from the defining equation (3.91).

For the candidate superpotentials $W^{(\alpha_\ell)}$, $\ell=1,2,3$, which have an expansion in fractional powers $q_y^{d_y/3}$ with $d_y\in\mathbb{Z}$, we did not find integral invariants with the multi-cover formula used in the other examples and in ref. [35]. It appears that only the numbers $n_{d_y,d_2}\cdot Z^{d_y}$, with Z a small power of 3, are integral. The solution to this problem might require a shift of the open string mirror map or a refinement of the multi-cover formula.

$q_y \backslash q_2$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	4	188	188	4	0	0	0	0	0
2	6	-68	5194	19024	5194	-68	6	0	0
3	24	-292	-3232	259524	3569704	3569704	259524	-3232	-292
4	112	-1660	-10996	-4092	13712184	555071696	1455120104	555071696	13712184
5	620	-10768	-42752	383424	-256440	695568492	74900481736	418921719720	418921719720
6	3732	-75468	-140150	4170468	6794752	-464516720	32227348614	9235136625472	97930146122188
7	24164	-556600	5648	37548816	24834800	-2671560544	-62352854944	991475402468	1066545645786456
8	164320	-4256460	7444296	318651284	-286806192	-20467318044	-282718652536	-7115509903004	-64593220192464
9	1162260	-33442800	114057840	2622725460	-7347237536	-170307380384	-1384203066912	-28014543398208	-915396773309428

Table 6: Symmetric disc invariants for the on-shell superpotentials $W^{(\alpha_0)}$ of the threefold $\mathbb{P}_{1,1,2,2,2}[8]$.

Extremal transition

At the singular locus $\phi^2 = 1$ ($z_2 = \frac{1}{4}$) there is an extremal transition to the mirror of the one-parameter model $\mathbb{P}^5[2,4]$ [9]. The large complex structure parameters $z^{(1)}$ of the one-parameter model and the two-parameter model are related by

$$z^{(1)} = \frac{1}{2}\, z_1 \tag{3.106}$$

To restrict the superpotential found in the two-parameter model to that of the one-parameter model we have to add as in ref. [35] an additional linear combination of bulk periods

$$\tilde{W}^{(\alpha_0)}(y, z_2) = W^{(\alpha_0)}(y, z_2) + 3F_1(y, z_2) + \frac{3}{2}F_2(y, z_2) \tag{3.107}$$

where $(2\pi i)^2\, F_1 = \partial_{\rho_1}\partial_{\rho_2} B_{\{l^1,l^2\}}|_{\rho_i=0}$ and $(2\pi i)^2\, F_2 = \partial_{\rho_1}^2 B_{\{l^1,l^2\}}|_{\rho_i=0}$. We then obtain

$$W^{(1,\alpha_0)}(z^{(1)}) = \tilde{W}^{\alpha_0}(8z^{(1)}, \tfrac{1}{4}) \tag{3.108}$$

Using the Picard-Fuchs operator of the one-parameter model

$$\mathcal{L}^{(1)} = \theta^4 - 8z^{(1)}(4\theta+1)(4\theta+2)(4\theta+3)(2\theta+1) \tag{3.109}$$

with $\theta = z^{(1)}\partial_{z^{(1)}}$ one obtains the inhomogeneous term

$$\mathcal{L}^{(1)} W^{(1,\alpha_0)} = \frac{224 z^{(1)}}{(2\pi i)^2}\left(1 + 272 z^{(1)} + \frac{285120 (z^{(1)})^2}{7} + 4925440 (z^{(1)})^3 + \ldots\right) \tag{3.110}$$

For the integer invariants we expect the following relation

$$\sum_{l=0}^{3k} n_{k,l}^{(\alpha_0)}(\mathbb{P}_{1,1,2,2,2}[8]) = n_k^{(\alpha_0)}(\mathbb{P}^5[2,4])\,. \tag{3.111}$$

However such a relation only emerges after the addition of an additional bulk period, again as in [35]

$$\tilde{W}^{(1,\alpha_0)}(z^{(1)}) = W^{(1,\alpha_0)}(z^{(1)}) - \frac{3}{2}F^{(1)}(z^{(1)})\,, \tag{3.112}$$

where $(2\pi i)^2\, F^{(1)} = \partial_\rho^2 B_{\{l^{(1)}\}}|_{\rho=0}$ with $l^{(1)} = (-4,-2\,|\,1,1,1,1,1,1)$. The invariants of $\tilde{W}^{(1,\alpha_0)}$ are given by

$$n_k^{(\alpha_0)} = 384, 29288, 7651456, 2592654592, 989035688064, \ldots \tag{3.113}$$

It would be interesting to also get a better understanding of the restriction of the superpotentials $W^{(\alpha_\ell)}, \ell = 1,2,3$, to the one-parameter model.

3.7 Degree 18 hypersurface in $\mathbb{P}_{1,2,3,3,9}$

This is a three parameter Calabi–Yau manifold with the charge vectors of the GLSM given by [53]:

	0	1	2	3	4	5	6	7
l^1	-6	-1	0	1	1	3	2	0
l^2	0	1	0	0	0	0	-2	1
l^3	0	0	1	0	0	0	1	-2

The hypersurface constraint is

$$P = x_1^{18} + x_2^9 + x_3^6 + x_4^6 + x_5^2 + \psi\, x_1x_2x_3x_4x_5 + \phi\, x_1^{12}x_2^3 + \chi\, x_1^6x_2^6\,, \tag{3.114}$$

where $\psi = z_1^{-1/6} z_2^{-2/9} z_3^{-1/9}$, $\phi = z_2^{-2/3} z_3^{-1/3}$ and $\chi = z_2^{-1/3} z_3^{-2/3}$. The orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with the weights

$$\mathbb{Z}_9 :\ g_1 = (1,-1,0,0,0)\,, \quad \mathbb{Z}_6 :\ g_2 = (1,0,-1,0,0)\,, \quad \mathbb{Z}_6 :\ g_3 = (1,0,0,-1,0)\,, \tag{3.115}$$

with $1 = \lambda_1^9 = \lambda_{2,3}^6$. In this geometry we consider the set of curves

$$C_\pm = \{x_3^3 = \pm i x_4^3\,,\ x_5 = \pm i x_1^9\,,\ x_2 = 0\}\,, \tag{3.116}$$

with the identifications $(+,+) \sim (-,-)$ and $(+,-) \sim (-,+)$ for the possible choices of sign under the orbifold group. The divisor

$$Q(\mathcal{D}) = x_3^6 + z_4 x_4^6 \tag{3.117}$$

leads by the now familiar steps to a GLSM for a K3 manifold with charges

	0	1	2	4	5	6	7
$\tilde{l}^1$	-6	-1	0	2	3	2	0
$\tilde{l}^2$	0	1	0	0	0	-2	1
$\tilde{l}^2$	0	0	1	0	0	1	-2

where the moduli of the surface are related to that of the Calabi-Yau threefold by $u_1 = -\frac{z_1}{z_4}(1-z_4)^2$, $u_2 = z_2$ and $u_3 = z_3$. The GLSM is again at a special codimension one locus in the moduli space, with the coefficient of the monomial $x_5 x_1^9$ set to zero. The solution

$$\pi(u) = \frac{c}{2}\, B_{\{\tilde{l}^1,\tilde{l}^2,\tilde{l}^3\}}(u_1,u_2,u_3;\tfrac{1}{2},0,0) = \frac{4c}{\pi}\sqrt{u_1} + \mathcal{O}(u_1^{3/2})\,, \tag{3.118}$$

	$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5	6
	1	1	1	0	0	0	0	0
	3	-27	-10	-10	-27	0	0	0
q_3^0	5	2840	-1629	2034	2034	-1629	2840	0
	7	-450807	523790	-501714	37970	37970	-501714	523790
	9	87114366	-143646335	151709190	-82679940	42724232	42724232	-82679940
	11	-18907171063	39698748864	-48496621950	38005868880	-25022027880	6124612608	6124612608

	$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5	6
	1	0	1	0	0	0	0	0
	3	0	-10	876	-10	0	0	0
q_3^1	5	0	-1629	-2520	595890	-2520	-1629	0
	7	0	523790	-3041532	702090	393040296	702090	-3041532
	9	0	-143646335	913643880	-2889725838	1131043400	248949858594	1131043400
	11	0	39698748864	-261938878740	899363170080	-2195675791704	998105927940	153662218213536

	$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5	6
	1	0	0	0	0	0	0	0
	3	0	0	-10	-10	0	0	0
q_3^2	5	0	0	2034	595890	595890	2034	0
	7	0	0	-501714	702090	1648025820	1648025820	702090
	9	0	0	151709190	-2889725838	691571574	2721112372690	2721112372690
	11	0	0	-48496621950	899363170080	-7230517669764	2911708467972	

	$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5	6
	1	0	0	0	0	0	0	0
	3	0	0	0	-27	0	0	0
q_3^3	5	0	0	0	2034	-2520	2034	0
	7	0	0	0	37970	393040296	1648025820	393040296
	9	0	0	0	-82679940	1131043400	2721112372690	8512061067684
	11	0	0	0	38005868880	-2195675791704		

Table 7: Disc invariants $\frac{1}{16} \cdot n_{d_1,d_2,d_3}$ for the on-shell superpotential $W^{(+)}$ of the threefold $\mathbb{P}_{1,2,3,3,9}[18]$.

vanishes at the critical locus $u_1 = 0$ and integrates to the superpotential

$$W^{(\pm)} = \mp \frac{c}{8} \; B_{\{l^1,l^2,l^3\}} \left(z_1, z_2, z_3; \tfrac{1}{2}, 0, 0\right) . \tag{3.119}$$

Using the multicover formula

$$\frac{W^{(\pm)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2} \sum_{k\, odd} \sum_{\substack{d_1\ odd \\ d_{2,3} \geq 0}} n^{(\pm)}_{d_1,d_2,d_3} \, \frac{q_1^{kd_1/2} \; q_2^{kd_2} \; q_3^{kd_3}}{k^2} \tag{3.120}$$

we obtain, for $c = 1$, the integer invariants in Tab. 7.

The closed-string type II compactification has a non-perturbative enhanced gauge symmetry with gauge group $G = SU(3)$ at the special values $t_2 = t_3 = 0$ of the closed-string moduli. The monodromy around the special locus acts as

$$m_1: \; t_1 \to t_1 + 2t_2, \; t_2 \to -t_2, \; t_3 \to t_2 + t_3, \qquad m_2: \; t_1 \to t_1, \; t_2 \to t_2 + t_3, \; t_3 \to -t_3,$$

and generates the Weyl group of $SU(3)$. The superpotential $W^{(\pm)}$ is a singlet under this group while the individual BPS states counted by the disc invariants are exchanged under the group action as $m_1: \; n_{d_1,d_2,d_3} \to n_{d_1,2d_1-d_2+d_3,d_3}$ and $m_2: \; n_{d_1,d_2,d_3} \to n_{d_1,d_2,d_2-d_3}$

The off-shell superpotentials are solutions of the following extended hypergeometric

system

$$
\begin{aligned}
\mathcal{L}_1 &= (\theta_2-\theta_1)(\theta_2-2\theta_3) - z_2(2\theta_1-2\theta_2+\theta_3-1)(2\theta_1-2\theta_2+\theta_3)\,,\\
\mathcal{L}_2 &= \theta_3(2\theta_1-2\theta_2+\theta_3) - z_3(\theta_2-2\theta_3-1)(\theta_2-2\theta_3)\,,\\
\mathcal{L}_3 &= \theta_3(\theta_2-\theta_1) - z_2z_3(2\theta_1-2\theta_2+\theta_3)(\theta_2-2\theta_3)\,,\\
\mathcal{L}_4 &= (\theta_1+\partial_y)(\theta_1-\partial_y)(2\theta_1-2\theta_2+\theta_3)\\
&\quad - 24z_1(6\theta_1+1)(6\theta_1+5)((4z_2-1)\theta_1+(3z_2z_3-4z_2+1)\theta_2+(2z_2-6z_2z_3)\theta_3)\,,\\
\mathcal{L}_5 &= (\theta_1+\partial_y)(\theta_1-\partial_y)(\theta_2-2\theta_3) - 8z_1z_2(6\theta_1+5)(6\theta_1+3)(6\theta_1+1)\,,\\
\mathcal{L}_6 &= \partial_y(\theta_1+\partial_y) + e^y\partial_y(\theta_1-\partial_y)\,,
\end{aligned}
\tag{3.121}
$$

where $y=\log(z_4)$.

To compute the inhomogeneous terms we note that the above differential operators are related to that of the Calabi–Yau threefold derived in [53] as

$$
\begin{aligned}
\mathcal{L}_a &= \mathcal{L}_a^{bulk}\,, \qquad a=1,2,3\,,\\
\mathcal{L}_4 &= \mathcal{L}_4^{bulk} - (2\theta_1-2\theta_2+\theta_3)\theta_4^2\,,\\
\mathcal{L}_5 &= \mathcal{L}_5^{bulk} - (\theta_2-2\theta_3)\theta_4^2\,.
\end{aligned}
\tag{3.122}
$$

to obtain from (3.118)

$$
\mathcal{L}_4^{bulk}W^{(\pm)} = \mp\frac{c}{\pi^2}\sqrt{z_1}\,, \qquad \mathcal{L}_a^{bulk}W^{(\pm)} = 0\,, a=1,2,3,5\,. \tag{3.123}
$$

Note that $\sqrt{z_1}=\psi^{-3}\phi$ is a rational function in terms of ψ and ϕ appearing in the hypersurface equation (3.114). The appearance of the square root is related to the non-trivial Greene-Plesser orbifold action on the defining equations (3.116) for the curves $C_\pm$.

As in the previous examples one may study the relation of the above brane geometry to (two and) one parameter configurations in a certain limit in the moduli. For $z_2=z_3=0$ the geometry approximates the non-compact Calabi–Yau of degree six discussed in App. A.2, explaining the relation $n_{k,0,0}=n_k^{[6]}$ between the invariants in Tab. 7 and Tab. 10.

At the point $t_2=t_3=0$ of $SU(3)$ gauge enhancement there is a transition to the one modulus Calabi-Yau $\mathbb{P}_{1,1,1,1,2,3}[3,6]$ [46], leading to the prediction

$$
\sum_{i,j} n_{k,i,j}(\mathbb{P}_{1,2,3,3,9}[18]) = n_k(\mathbb{P}_{1,1,1,1,2,3}[6,3])\,,
$$

where the first numbers are

$$
\frac{1}{16}n_k = 3,\ 735,\ 1791060,\ 6117294147,\ 25579918417320. \tag{3.124}
$$

The superpotential of the one parameter model is the solution of the inhomogeneous Picard-Fuchs equation

$$
\mathcal{L}^{bulk}W = \frac{3}{(2\pi i)^2}\sqrt{z_1}\,, \qquad \mathcal{L}^{bulk} = \theta_1^4 - 36z_1(3\theta_1+1)(3\theta_1+2)(6\theta_1+5)(6\theta_1+1)\ .
$$

3.8 Degree 12 hypersurface in $\mathbb{P}_{1,2,3,3,3}$

This example is very similar to the hypersurface in $\mathbb{P}_{1,2,3,3,9}$ studied above. The charge vectors of the GLSM given by [53]:

	0	1	2	3	4	5	6	7
l^1	-4	-1	0	1	1	1	2	0
l^2	0	1	0	0	0	0	-2	1
l^3	0	0	1	0	0	0	1	-2

The hypersurface constraint is

$$P = x_1^{12} + x_2^6 + x_3^4 + x_4^4 + x_5^4 + \psi\, x_1 x_2 x_3 x_4 x_5 + \phi\, x_1^8 x_2^2 + \chi\, x_1^4 x_2^4\,, \tag{3.125}$$

where $\psi = z_1^{-1/4} z_2^{-1/3} z_3^{-1/6}$, $\phi = z_2^{-2/3} z_3^{-1/3}$ and $\chi = z_2^{-1/3} z_3^{-2/3}$. The orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with the weights

$$\mathbb{Z}_6 :\ g_1 = (1,-1,0,0,0)\,, \quad \mathbb{Z}_4 :\ g_2 = (1,0,-1,0,0)\,, \quad \mathbb{Z}_4 :\ g_3 = (1,0,0,-1,0)\,, \tag{3.126}$$

with $1 = \lambda_1^6 = \lambda_{2,3}^4$. In this geometry we consider the set of curves

$$C_\pm = \{x_3^2 = \pm i x_4^2\ ,\ x_5^2 = \pm i x_1^6\ ,\ x_2 = 0\}\,, \tag{3.127}$$

with the identifications $(+,+) \sim (-,-)$ and $(+,-) \sim (-,+)$ for the possible choices of sign under the orbifold group. The divisor

$$Q(\mathcal{D}) = x_3^4 + z_4 x_4^4 \tag{3.128}$$

leads by the now familiar steps to a GLSM for a K3 manifold with charges

	0	1	2	4	5	6	7
$\tilde{l}^1$	-4	-1	0	2	1	2	0
$\tilde{l}^2$	0	1	0	0	0	-2	1
$\tilde{l}^2$	0	0	1	0	0	1	-2

where the moduli of the surface are related to that of the Calabi-Yau threefold by $u_1 = -\frac{z_1}{z_4}(1-z_4)^2$, $u_2 = z_2$ and $u_3 = z_3$. The GLSM is again at a special co-dimension one locus in the moduli space. The solution

$$\pi = \frac{c}{2} B_{\{\tilde{l}^1,\tilde{l}^2,\tilde{l}^3\}}(u_1,u_2,u_3;\tfrac{1}{2},0,0) = \frac{2c}{\pi}\sqrt{u_1} + \mathcal{O}(u_1^{3/2})\,, \tag{3.129}$$

vanishes at the critical locus $u_1 = 0$ and integrates to the superpotential

$$W^{(\pm)} = \mp\frac{c}{8}\, B_{\{l^1,l^2,l^3\}}\left(z_1,z_2,z_3;\tfrac{1}{2},0,0\right)\ . \tag{3.130}$$

Using the multicover formula (3.120) we obtain, for $c = 1$ the integer invariants in Tab. 8.

The closed-string type II compactification has a non-perturbative enhanced gauge symmetry with gauge group $G = SU(3)$ at $t_2 = t_3 = 0$. The monodromy around this special locus acts as

$$m_1 :\ t_1 \to t_1,\ t_2 \to -t_2,\ t_3 \to t_2 + t_3, \qquad m_2 :\ t_1 \to t_1 + t_3,\ t_2 \to t_2 + t_3,\ t_3 \to -t_3,$$

q_3^0

$q_1^{1/2}\backslash q_2$	0	1	2	3	4	5
1	1	0	0	0	0	0
3	-3	0	0	0	0	0
5	40	0	0	0	0	0
7	-847	0	0	0	0	0
9	21942	0	0	0	0	0
11	-640431	0	0	0	0	0

q_3^1

$q_1^{1/2}\backslash q_2$	0	1	2	3	4	5
1	1	1	0	0	0	0
3	-2	-2	0	0	0	0
5	-45	-45	0	0	0	0
7	1750	1750	0	0	0	0
9	-61551	-61551	0	0	0	0
11	2233440	2233440	0	0	0	0

q_3^2

$q_1^{1/2}\backslash q_2$	0	1	2	3	4	5
1	0	0	0	0	0	0
3	-2	108	-2	0	0	0
5	50	-56	50	0	0	0
7	-1962	-11196	-1962	0	0	0
9	86630	439560	86630	0	0	0
11	-3842790	-16939860	-3842790	0	0	0

q_3^3

$q_1^{1/2}\backslash q_2$	0	1	2	3	4	5
1	0	0	0	0	0	0
3	-3	-2	-2	-3	0	0
5	50	11090	11090	50	0	0
7	506	1634	1634	506	0	0
9	-67884	-1577166	-1577166	-67884	0	0
11	4125840	66691520	66691520	4125840	0	0

q_3^4

$q_1^{1/2}\backslash q_2$	0	1	2	3	4	5
1	0	0	0	0	0	0
3	0	0	0	0	0	0
5	-45	-56	11090	-56	-45	0
7	506	1127464	4423692	1127464	506	0
9	28776	517288	46134	517288	28776	0
11	-3030696	-185400024	-566257044	-185400024	-3030696	0

q_3^5

$q_1^{1/2}\backslash q_2$	0	1	2	3	4	5
1	0	0	0	0	0	0
3	0	0	0	0	0	0
5	40	-45	50	50	-45	40
7	-1962	1634	4423692	4423692	1634	-1962
9	28776	111025794	1085027250	1085027250	111025794	28776
11	1030368	74577268	129171092	129171092	74577268	1030368

Table 8: Disc invariants $\frac{1}{8} \cdot n_{d_1,d_2,d_3}$ for the on-shell superpotential $W^{(+)}$ of the threefold $\mathbb{P}_{1,2,3,3,3}[12]$.

and generates the Weyl group of $SU(3)$. The superpotential $W^{(\pm)}$ is a singlet under this group while the individual BPS states counted by the disc invariants are exchanged under the group action as $m_1 : \; n_{d_1,d_2,d_3} \to n_{d_1,-d_2+d_3,d_3}$ and $m_2 : \; n_{d_1,d_2,d_3} \to n_{d_1,d_2,d_1+d_2-d_3}$.

The off-shell superpotentials are solutions of the following extended hypergeometric system

$$
\begin{aligned}
\mathcal{L}_1 &= (\theta_2 - \theta_1)(\theta_2 - 2\theta_3) - z_2(2\theta_1 - 2\theta_2 + \theta_3 - 1)(2\theta_1 - 2\theta_2 + \theta_3)\,, \\
\mathcal{L}_2 &= \theta_3(2\theta_1 - 2\theta_2 + \theta_3) - z_3(\theta_2 - 2\theta_3 - 1)(\theta_2 - 2\theta_3)\,, \\
\mathcal{L}_3 &= \theta_3(\theta_2 - \theta_1) - z_2 z_3(2\theta_1 - 2\theta_2 + \theta_3)(\theta_2 - 2\theta_3)\,, \\
\mathcal{L}_4 &= (\theta_1 + \partial_y)(\theta_1 - \partial_y)(2\theta_1 - 2\theta_2 + \theta_3) \\
&\qquad - 8z_1(4\theta_1 + 3)(4\theta_1 + 1)((4z_2 - 1)\theta_1 + (3z_2 z_3 - 4z_2 + 1)\theta_2 - (6z_2 z_3 - 2z_2)\theta_3)\,, \\
\mathcal{L}_5 &= (\theta_1 + \partial_y)(\theta_1 - \partial_y)(\theta_2 - 2\theta_3) - 4z_1 z_2(4\theta_1 + 3)(4\theta_1 + 2)(4\theta_1 + 1)\,, \\
\mathcal{L}_6 &= \partial_y(\theta_1 + \partial_y) + e^y \partial_y(\theta_1 - \partial_y)\,,
\end{aligned}
\tag{3.131}
$$

where $y = \log(z_4)$.

To compute the inhomogeneous terms we note that the above differential operators are related to that of the Calabi–Yau threefold as

$$\begin{aligned} \mathcal{L}_a &= \mathcal{L}_a^{bulk}\,, \quad a = 1, 2, 3\,, \\ \mathcal{L}_4 &= \mathcal{L}_4^{bulk} - (2\theta_1 - 2\theta_2 + \theta_3)\theta_4^2\,, \\ \mathcal{L}_5 &= \mathcal{L}_5^{bulk} - (\theta_2 - 2\theta_3)\theta_4^2\,. \end{aligned} \tag{3.132}$$

Then we obtain from eq. (3.129)

$$\mathcal{L}_4^{bulk} W^{(\pm)} = \mp\frac{c}{2\pi^2} \cdot \sqrt{z_1}\,, \qquad \mathcal{L}_a^{bulk} W^{(\pm)} = 0\,, a = 1, 2, 3, 5\,, \tag{3.133}$$

where, similarly as in the previous example, $\sqrt{z_1} = \psi^{-2}\phi$ is a rational function in ψ and ϕ, and the appearance of the square root is related to the non-trivial action of the Greene-Plesser orbifold group on the defining equations for the curves $C_\pm$.

In the limit $z_2 = z_3 = 0$ we can again make contact with a non-compact Calabi-Yau. Here it is the degree four hypersurface discussed in App. A.2. This explains the relation $n_{k,0,0} = n_k^{[4]}$ between the invariants in Tab. 8 and Tab. 10.

At the point $t_2 = t_3 = 0$ of $SU(3)$ gauge enhancement there is a transition to the one modulus Calabi–Yau $\mathbb{P}_{1,1,1,1,1,2}[3,4]$ [46], leading to the prediction $\sum_{i,j} n_{k,i,j}(\mathbb{P}_{1,2,3,3,3}[12]) = n_k(\mathbb{P}_{1,1,1,1,1,2}[3,4])$, with the first invariants being

$$\frac{1}{8}\, n_k = 3,\ 87,\ 33252,\ 16628907,\ 10149908544,\ 6979959014559, 5196581251886028\,. \tag{3.134}$$

The superpotential is a solution of the inhomogeneous Picard-Fuchs equation of the one modulus problem

$$\mathcal{L}^{bulk} W = -\frac{3}{8\pi^2}\sqrt{z_1}\,, \qquad \mathcal{L}^{bulk} = \theta_1^4 - 12z_1(3\theta_1+1)(3\theta_1+2)(4\theta_1+1)(4\theta_1+3)\ .$$

4. Conclusions and outlook

In this work we studied off-shell brane superpotentials for four-dimensional type II/F-theory compactifications depending on several open-closed deformations as well as their specialization to the on-shell values in the open-string direction. Mathematically the two potentials are respectively related to the integral period integrals on the (relative) cohomology group defined by the family of branes [6, 7, 15, 16, 17, 20, 26], which depend on both open and closed deformations, and the so-called normal functions, depending only on closed-string moduli [35, 13]. Both objects can be studied Hodge theoretically by computing the variation of Hodge structure on the relevant (co-)homology fibers over the open-closed-string deformation space $\mathcal{M}$. Ultimately, this determines the superpotential as a particular solution of a system of generalized GKZ type differential equations determined by the integral (relative) homology class of the brane.

The D-brane superpotentials computed in this way are relevant in different contexts. From the phenomenological point of view, the superpotential determines the vacuum structure of four-dimensional F-theory compactifications. The complicated structure of the

superpotential for this class of compactifications, described by infinite generalized hypergeometric series, should be contrasted with the simple structure of F-theory superpotentials in other classes of compactifications, as e.g. in refs. [54, 55]. These hypergeometric series have sometimes a dual interpretation as D-instanton corrections and heterotic world-sheet corrections [27], and the rich structure of non-perturbative corrections to the brane superpotential should lead to interesting hierarchies of masses and couplings in the low-energy effective theory.

As shown in ref. [27], the solutions to the generalized GKZ system representing the F-theory superpotential do not only capture the superpotentials of dual Calabi-Yau threefold compactifications, but more generally of type II and heterotic compactifications on generalized Calabi-Yau manifolds of complex dimension three.[22] This offers a powerful tool to study more generally the vacuum structure of phenomenologically interesting F-theory/type II/heterotic compactifications. It would be interesting to apply the Hodge theoretic approach described in this paper to examples of phenomenologically motivated F-theory scenarios, as described e.g. in refs. [57, 58, 59].[23] In the search for vacua, the step of passing from relative periods depending on open and closed-string deformations to normal functions depending only on closed-string moduli provides a natural split in the minimization process, which should be helpful in a regime of small string coupling. On the other hand, this distinction between closed and open-string moduli disappears away from this decoupling limit, for finite string coupling, where the two types of fields mix in a way determined by a certain degeneration of the F-theory fourfold described in [17, 27].

A complementary aspect of the B-type superpotentials considered in this paper is the prediction of A model disc invariants by open-closed mirror symmetry. For almost flat open-string directions (characterized by a generalized large complex structure point in the open-closed deformation space [17, 26]), already the off-shell superpotential computed by the relative period integral has an A model expansion in closed- and open-string parameters, leading to predictions for integral Ooguri-Vafa invariants [17, 30, 26, 27]. In the present work we instead concentrated on the critical points of the type studied in refs. [12, 13, 14, 33, 34, 61], where the A model expansion emerges only after integrating out the open-string directions. The on-shell computations of refs. [12, 13, 14] are conceptually well understood and provided the first examples of open-string mirror symmetry in compact Calabi–Yau. Our main motivation to study the type of critical points accessible also in the on-shell formalism was to gain a better understanding of the minimization in the open-string direction, which relates the on-shell computation to the off-shell framework of refs. [6, 7, 15, 16, 17]. On the B model side, the relation is provided by the connection between integral relative period integrals and normal functions described in sect. 2. An important datum in this correspondence is the period vector on the surface, that is the brane 4-cycle. It classifies the D-brane vacua by the vanishing condition (2.7) and determines the inhomogeneous term in the Picard-Fuchs equation for the normal function.

In the relative cohomology approach of refs. [6, 7, 15, 16, 17], the open-string de-

[22]The first examples of dual compactifications of this type were given in ref. [28]. See also refs. [29, 56] for related works and examples.

[23]See also ref. [60], for a recent review on this subject, and further references therein.

formations are off-shell yet one avoids working in string field theory by perturbing the unobstructed F-theory moduli space associated with the family of surfaces $\mathcal{D}$ by a probe brane representing an element in $H_2(\mathcal{D})$. This leads to well-defined *finite dimensional* off-shell deformation spaces associated with a particular parametrization by 'light' fields in the superpotential. The parametrization of off-shell deformations is adapted to the topological string and leads to a definition of off-shell mirror maps and off-shell invariants consistent with expectations. By the general arguments of sect. 2.2, different parametrizations are bound to fit together in an consistent way, as is explicitly demonstrated in some of the examples, where we parametrized the off-shell superpotentials by different choices of open-string deformation parameters. This means starting from a given supersymmetric configuration, we compare different off-shell deformation directions in the infinite-dimensional open-closed deformation space, and we find that the obtained on-shell tensions are independent from the chosen off-shell directions.[24] This is a gratifying result as the on-shell domain wall tensions should not depend on the details of integrating out the heavy modes.

The relative cohomology approach to open-closed deformations has successfully passed other non-trivial checks [20, 31]. In leading order the computed off-shell superpotentials are compatible with derivations of effective superpotentials using open-string worldsheet and matrix factorization techniques [62, 63, 64, 31, 65, 66, 67]. Beyond leading order, however, the discussed off-shell superpotentials predict in the context of type II theories higher order open-closed CFT correlators, which (at present) are difficult to compute by other means.

There are many other open questions that need further exploration. For examples with a single open-string deformation a detailed analysis of the Hodge structure of the K3 surface, equivalent to the subsystem defined by the Hodge structure on the surface $\mathcal{D}$, might be rewarding. In this work we explained how the analyzed supersymmetric domain wall tensions arise at enhancement points of the Picard lattice in the K3 moduli space. The leading term of the K3 periods near these specially symmetric points is a rational function in the deformations z and the roots α of the defining equation. As argued in sect. 3.1, the global symmetry seems to be related to the discrete symmetry in the A-type brane in the mirror A-model configuration. It would be interesting to study in detail the structure of Picard lattice enhancement loci in order to systematically explore the web of $\mathcal{N} = 1$ domain wall tensions in Calabi-Yau threefolds. Such an analysis potentially sheds light on the global structure of $\mathcal{N} = 1$ superpotentials (see e.g. refs. [68]), and should be related to the wall-crossing phenomena described in refs. [69, 70, 71].

In this work we have focused on a single open-string deformation. Then the subsystem of the extended hypergeometric GKZ system, which governs the open deformations, describes the periods of an isogenic K3 surface. The presented techniques are directly applicable also to examples with several open deformations [72]. Then the subsystem geometry is not anymore governed by K3 periods but instead by the periods of a complex surface of a higher geometric genus. Exploring such examples is technically more challenging but new phenomena and interesting structures, like non-commutativity in the open-string sector, are likely to emerge. A related question in this context is the contribution from D-instanton

[24] See ref. [20] for an earlier example of this kind.

corrections, which are also computed by GKZ system for the F-theory compactification [27]. It would be very interesting to connect these results to the recent progress in computing D-brane instantons by different methods [73, 74, 75, 76, 77].

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A. Appendix

A.1 Toric hypersurfaces for type II and F-theory compactifications

In the framework of [10] a mirror pair (X^*, X) of Calabi-Yau threefolds is given as a pair of hypersurfaces defined in two toric ambient spaces (V^*, V) as follows. The toric varieties (V^*, V) are associated to the fans $(\Sigma(\Delta^*), \Sigma(\Delta))$ obtained from the set of cones over the faces of two dual reflexive polyhedra (Δ^*, Δ). The polyhedron Δ^* is the convex hull of p integral points $\nu_i^\star \in \mathbb{Z}^5 \in \mathbb{R}^5$ lying in a hyperplane of distance one to the origin and Δ is the dual polyhedron with integral points ν_i. The mirror 3-folds X are defined in V as the zero locus of the hypersurface constraint

$$P = \sum_{i=0}^{p-1} a_i \prod_{k=1}^{4} X_k^{\nu_{i,k}^*} .$$

Here the X_k, $k = 1, \ldots, 4$ are coordinates on an open torus $(\mathbb{C}^*)^4 \subset V$ and a_i are complex parameters which determine the complex structure of X. Alternatively, one may write the hypersurface in homogeneous coordinates x_j on the toric ambient space as

$$P = \sum_{i=0}^{p-1} a_i \prod_{\nu_j \in \Delta} x_j^{<\nu_j, \nu_i^*>+1} . \tag{A.1}$$

Keeping only the coordinates x_i associated with the vertices of Δ in the product on the right hand side, one obtains the simplified expression used in the text, e.g. eq.(3.2) in the first example.

The integral points $\nu_i^\star$ of $\Delta^\star$ fulfill a set of linear relations $\sum_{i=0}^{p-1} l_i^a \nu_i^\star = 0$ specified by $M = h^{(1,1)}(X^*) = h^{(2,1)}(X)$ vectors l^a, $a = 1, \ldots, M$ with integral entries, given e.g. in (3.1) for the first example. The vectors l^a represent the $U(1)$ charges of the two-dimensional

fields in the GLSM associated to X [78]. The first index $i = 0$ refers to the single interior point of Δ, which corresponds to the distinguished field of negative charge that multiplies the hypersurface constraint in the two dimensional superpotential.

The open-string sector for the compactification with B-type branes on X is captured by the family of hypersurfaces $\mathcal{D}$ defined as the complete intersections $\mathcal{D}: \{P(X) = 0\} \cap \{Q(\mathcal{D}) = 0\}$ in V [4, 7, 17, 20]. Locally, one may write $Q(D)$ as

$$Q(\mathcal{D}) = \sum_{i=p}^{p+p'-1} a_i X_k^{\nu_i^\star,k} , \tag{A.2}$$

where the right hand side defines p' additional (not necessarily integral) vertices $\nu_i^\star$ associated with the monomials in $Q(\mathcal{D})$. The coefficients $a_i,\ i \geq p$ are complex parameters of a family of embeddings $\mathcal{D} \hookrightarrow X$ for a fixed set of monomials in $Q(\mathcal{D})$ and determine a point on the fiber $\hat{\mathcal{M}}$ of the deformation space

$$\hat{\mathcal{M}}(\hat{z}) \longrightarrow \mathcal{M} \xrightarrow{\ \pi\ } \mathcal{M}_{CS}(z) \ . \tag{A.3}$$

The combined data for the closed and the open-string sector can be expressed in terms of extended vertices $\bar{\nu}^\star{}_i$, which makes contact to the F-theory compactification on a fourfold dual to the brane geometry [16, 20, 26, 27]. To this end, consider the set of extended vertices

$$\bar{\nu}^\star{}_i = \begin{cases} (\nu_i^\star, 0) & i = 0, ..., p-1 \\ (\nu_i^\star, 1) & i = p, ..., p+p'-1 \end{cases} ,$$

which determine the (ordered) monomials in (A.1) and (A.2). Define the set $L = \{\underline{l}\}$ as the set of solutions to the equations

$$\sum_{i=0}^{p+p'-1} \underline{l}_i \bar{\nu}^\star{}_i = 0, \qquad \sum_{i=0}^{p-1} \underline{l}_i = 0 = \sum_{i=p}^{p+p'-1} \underline{l}_i \, . \tag{A.4}$$

At this point, the $\bar{\nu}^\star{}_i$ for $i \geq p$ are defined only up to an overall shift $\nu_i \to \bar{\nu}^\star{}_i + (\mu, 0)$ for a constant four-vector μ (corresponding to multiplication of Q by an overall factor), but this shift is not relevant in (A.4) because of the last condition. For the generators of L one may choose the charge vectors of the closed-string GLSM extended by p' zeros to the left and in addition $p'-1$ vectors describing relations between the monomials in $Q(\mathcal{D})$ and those in P.

$$\underline{l}^a = \begin{cases} (\underline{l}^a, 0, ..., 0) & i = 1, ..., M \\ (...) & a = M+1, ..., M+p'-1 \end{cases} . \tag{A.5}$$

From these vectors one obtains the parameters

$$z_a = (-)^{\underline{l}_0^a} \prod_i a_i^{\underline{l}_i^a}, \qquad a = 1, ..., h^{2,1}(X) + p' - 1 \, , \tag{A.6}$$

invariant under the torus action. For $a \leq h^{2,1}(X)$, these are the coordinates on the base $\mathcal{M}_{CS}$ and the z_a for $a > h^{2,1}(X)$ describe open-string deformations. If the vertices $\bar{\nu}^\star{}_i$

satisfy appropriate extra conditions discussed in [16, 17, 26], the z_a define local coordinates near an open-string generalization of a large complex structure point $\mathcal{P}$ in $\mathcal{M}$, where the superpotential has an A model expansion in Ooguri–Vafa invariants .

The extended vertices $\bar{\nu}^\star{}_i$ for the brane geometry on the threefold X define an extended polyhedron $\underline{\Delta}^\star$ of one dimension higher, which can be associated to mirror pair of non-compact Calabi–Yau fourfolds $(X_4^{*,\sharp}, X_4^\sharp)$ [6, 7, 16, 26]. M/F-theory compactification on $X_4^\sharp$ gives a dual compactification without branes but with flux, related to the brane compactification on the threefold X by open-closed duality [19, 17, 20]. Under duality, the RR brane (and flux) superpotential on the threefold X maps to the leading order term of the GVW superpotential on $X_4^\sharp$ in the expansion (2.3) in g_s, that is

$$W_{GVW}(X_4^\sharp) = \sum_\Sigma N_\Sigma(G)\, \underline{\Pi}_\Sigma(z, \hat{z}) \ . \tag{A.7}$$

This open-closed duality at $g_s = 0$ extends to a full string duality between the brane compactification on X and F-theory compactification on a compact fourfold X_4 [17, 27]. The details of the compactification capture the coupling of the brane to the global geometry and affect only the higher order terms in g_s, but not the disc invariants.

We hence restrict to report the polyhedra for the non-compact 4-folds $X_4^\sharp$ below, which determine the leading order superpotential by eq. (A.7). In the following table we collect the (extended) points $\bar{\nu}^\star{}_i$ for the brane geometry in the examples and the dual vertices ν_i defining the homogeneous coordinates used in the text via eq. (A.1). The points $\nu_i^\star$ for the threefold X are given by the subset of the $\bar{\nu}^\star{}_i$ with vanishing last entry, $\bar{\nu}^\star{}_i = (\nu_i^\star, 0)$.

	$\bar{\Delta}^\star(X_4^\sharp) \ \supset \ \Delta^\star(X)$	$\Delta(X^\star)$
$\mathbb{P}_{1,2,2,3,4}[12]$	$\bar{\nu}^\star{}_1 = (\ 2,\ 2,\ 3,\ 4;\ 0)$	$\nu_1 = (\ 1,\ 1,\ 1,\ 1)$
	$\bar{\nu}^\star{}_2 = (-1,\ 0,\ 0,\ 0;\ 0)$	$\nu_2 = (-5,\ 1,\ 1,\ 1)$
	$\bar{\nu}^\star{}_3 = (\ 0,-1,\ 0,\ 0;\ 0)$	$\nu_3 = (\ 1,-5,\ 1,\ 1)$
	$\bar{\nu}^\star{}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\nu_4 = (\ 1,\ 1,-3,\ 1)$
	$\bar{\nu}^\star{}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\nu_5 = (\ 1,\ 1,\ 1,-2)$
	$\bar{\nu}^\star{}_6 = (\ 1,\ 1,\ 1,\ 2;\ 0)$	
$X_4^\sharp(\mathcal{D}_1)$	$\bar{\nu}^\star{}_7 = (\nu_2^\star;1),\ \bar{\nu}^\star{}_8 = (\nu_3^\star;1)$	
$X_4^\sharp(\mathcal{D}_2)$	$\bar{\nu}^\star{}_7 = (\nu_4^\star;1),\ \bar{\nu}^\star{}_8 = (\nu_6^\star;1)$	
$\mathbb{P}_{1,2,2,2,7}[14]$	$\bar{\nu}^\star{}_1 = (\ 2,\ 2,\ 2,\ 7;\ 0)$	$\nu_1 = (\ 1,\ 1,\ 1,\ 1)$
	$\bar{\nu}^\star{}_2 = (-1,\ 0,\ 0,\ 0;\ 0)$	$\nu_2 = (-6,\ 1,\ 1,\ 1)$
	$\bar{\nu}^\star{}_3 = (\ 0,-1,\ 0,\ 0;\ 0)$	$\nu_3 = (\ 1,-6,\ 1,\ 1)$
	$\bar{\nu}^\star{}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\nu_4 = (\ 1,\ 1,-6,\ 1)$
	$\bar{\nu}^\star{}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\nu_5 = (\ 1,\ 1,\ 1,-1)$
	$\bar{\nu}^\star{}_6 = (\ 1,\ 1,\ 1,\ 3;\ 0)$	
$X_4^\sharp(\mathcal{D}_1)$	$\bar{\nu}^\star{}_7 = (\nu_3^\star;1),\ \bar{\nu}^\star{}_8 = (\nu_4^\star;1)$	
$X_4^\sharp(\mathcal{D}_2)$	$\bar{\nu}^\star{}_7 = (\nu_5^\star;1),\ \bar{\nu}^\star{}_8 = (\nu_6^\star;1)$	
$\mathbb{P}_{1,1,1,6,9}[18]$	$\bar{\nu}^\star{}_1 = (\ 1,\ 1,\ 6,\ 9;\ 0)$	$\nu_1 = (\ 1,\ 1,\ 1,\ 1)$
	$\bar{\nu}^\star{}_2 = (-1,\ 0,\ 0,\ 0;\ 0)$	$\nu_2 = (-17,\ 1,\ 1,\ 1)$
	$\bar{\nu}^\star{}_3 = (\ 0,-1,\ 0,\ 0;\ 0)$	$\nu_3 = (\ 1,-17,\ 1,\ 1)$
	$\bar{\nu}^\star{}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\nu_4 = (\ 1,\ 1,-2,\ 1)$
	$\bar{\nu}^\star{}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\nu_5 = (\ 1,\ 1,\ 1,-1)$
	$\bar{\nu}^\star{}_6 = (\ 0,\ 0,\ 2,\ 3;\ 0)$	
$X_4^\sharp(D)$	$\bar{\nu}^\star{}_7 = (\nu_1^\star;1),\ \bar{\nu}^\star{}_8 = (\nu_2^\star;1)$	
$\mathbb{P}_{1,1,1,3,3}[9]$	$\bar{\nu}^\star{}_1 = (-1,\ 0,\ 1,\ 1;\ 0)$	$\nu_1 = (-6,\ 3,\ 1,\ 1)$
	$\bar{\nu}^\star{}_2 = (\ 0,-1,\ 1,\ 1;\ 0)$	$\nu_2 = (\ 3,-6,\ 1,\ 1)$
	$\bar{\nu}^\star{}_3 = (\ 1,\ 1,\ 1,\ 1;\ 0)$	$\nu_3 = (\ 3,\ 3,\ 1,\ 1)$
	$\bar{\nu}^\star{}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\nu_4 = (\ 0,\ 0,-2,\ 1)$
	$\bar{\nu}^\star{}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\nu_5 = (\ 0,\ 0,\ 1,-2)$
	$\bar{\nu}^\star{}_6 = (\ 0,\ 0,\ 1,\ 1;\ 0)$	
$X_4^\sharp(D)$	$\bar{\nu}^\star{}_7 = (\nu_1^\star;1),\ \bar{\nu}^\star{}_8 = (\nu_2^\star;1)$	
$\mathbb{P}_{1,1,2,2,6}[12]$	$\bar{\nu}^\star{}_1 = (\ 1,\ 2,\ 2,\ 6;\ 0)$	$\nu_1 = (\ 1,\ 1,\ 1,\ 1)$
	$\bar{\nu}^\star{}_2 = (-1,\ 0,\ 0,\ 0;\ 0)$	$\nu_2 = (-11,\ 1,\ 1,\ 1)$
	$\bar{\nu}^\star{}_3 = (\ 0,-1,\ 0,\ 0;\ 0)$	$\nu_3 = (\ 1,-5,\ 1,\ 1)$
	$\bar{\nu}^\star{}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\nu_4 = (\ 1,\ 1,-5,\ 1)$
	$\bar{\nu}^\star{}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\nu_5 = (\ 1,\ 1,\ 1,-1)$
	$\bar{\nu}^\star{}_6 = (\ 0,\ 1,\ 1,\ 3;\ 0)$	
$X_4^\sharp(\mathcal{D})$	$\bar{\nu}^\star{}_7 = (\nu_5^\star;1),\ \bar{\nu}^\star{}_8 = (0,0,0,0;1)$	
$\mathbb{P}_{1,1,2,2,2}[8]$	$\bar{\nu}^\star{}_1 = (\ 1,\ 2,\ 2,\ 2;\ 0)$	$\nu_1 = (\ 1,\ 1,\ 1,\ 1)$
	$\bar{\nu}^\star{}_2 = (-1,\ 0,\ 0,\ 0;\ 0)$	$\nu_2 = (-7,\ 1,\ 1,\ 1)$
	$\bar{\nu}^\star{}_3 = (\ 0,-1,\ 0,\ 0;\ 0)$	$\nu_3 = (\ 1,-3,\ 1,\ 1)$
	$\bar{\nu}^\star{}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\nu_4 = (\ 1,\ 1,-3,\ 1)$
	$\bar{\nu}^\star{}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\nu_5 = (\ 1,\ 1,\ 1,-3)$
	$\bar{\nu}^\star{}_6 = (\ 0,\ 1,\ 1,\ 1;\ 0)$	
$X_4^\sharp(\mathcal{D})$	$\bar{\nu}^\star{}_7 = (\nu_5^\star;1),\ \bar{\nu}^\star{}_8 = (0,0,0,0;1)$	

	$\bar{\Delta}^\star(X_4^\sharp)\ \supset\ \Delta^\star(X)$	$\Delta(X^\star)$
$\mathbb{P}_{1,2,3,3,9}[18]$	$\bar{\nu}^\star{}_1 = (2,3,3,9;0)$	$\nu_1 = (1,1,1,1)$
	$\bar{\nu}^\star{}_2 = (-1,0,0,0;0)$	$\nu_2 = (-8,1,1,1)$
	$\bar{\nu}^\star{}_3 = (0,-1,0,0;0)$	$\nu_3 = (1,-5,1,1)$
	$\bar{\nu}^\star{}_4 = (0,0,-1,0;0)$	$\nu_4 = (1,1,-5,1)$
	$\bar{\nu}^\star{}_5 = (0,0,0,-1;0)$	$\nu_5 = (1,1,1,-1)$
	$\bar{\nu}^\star{}_6 = (1,2,2,6;0)$	
	$\bar{\nu}^\star{}_7 = (0,1,1,3;0)$	
$X_4^\sharp(\mathcal{D})$	$\bar{\nu}^\star{}_8 = (\nu_3^\star;1),\ \bar{\nu}^\star{}_9 = (\nu_4^\star;1)$	
$\mathbb{P}_{1,2,3,3,3}[12]$	$\bar{\nu}^\star{}_1 = (2,3,3,3;0)$	$\nu_1 = (1,1,1,1)$
	$\bar{\nu}^\star{}_2 = (-1,0,0,0;0)$	$\nu_2 = (-5,1,1,1)$
	$\bar{\nu}^\star{}_3 = (0,-1,0,0;0)$	$\nu_3 = (1,-3,1,1)$
	$\bar{\nu}^\star{}_4 = (0,0,-1,0;0)$	$\nu_4 = (1,1,-3,1)$
	$\bar{\nu}^\star{}_5 = (0,0,0,-1;0)$	$\nu_5 = (1,1,1,-3)$
	$\bar{\nu}^\star{}_6 = (1,2,2,2;0)$	
	$\bar{\nu}^\star{}_7 = (0,1,1,1;0)$	
$X_4^\sharp(\mathcal{D})$	$\bar{\nu}^\star{}_8 = (\nu_3^\star;1),\ \bar{\nu}^\star{}_9 = (\nu_4^\star;1)$	

Table 9: Vertices of the toric polyhedra for the threefolds for type II compactification and the non-compact limit of the F-theory fourfolds.

A.2 From three-chains to Abel-Jacobi maps on the elliptic curve

In some of the examples considered in sect. 3, the domain wall tensions can be directly related to the Abel-Jacobi map on an elliptic curve in a certain limit in the moduli. This gives a check on the normalization obtained from the geometric surface periods. To this end, we consider the non-compact Calabi-Yau manifolds $X^\flat$ of ref. [48]

$$\begin{aligned}
\mathbb{P}^4_{3,2,1,1,-1}[6]: &\quad P = y_1^2 + y_2^3 + y_3^6 + y_4^6 + \frac{1}{y_5^6} + \hat{\psi}\, y_1y_2y_3y_4y_5\ , \quad z = \hat{\psi}^{-6}\ ,\\
\mathbb{P}^4_{2,1,1,1,-1}[4]: &\quad P = y_1^2 + y_2^4 + y_3^4 + y_4^4 + \frac{1}{y_5^4} + \hat{\psi}\, y_1y_2y_3y_4y_5\ , \quad z = \hat{\psi}^{-4}\ ,\\
\mathbb{P}^4_{1,1,1,1,-1}[3]: &\quad P = y_1^3 + y_2^3 + y_3^3 + y_4^3 + \frac{1}{y_5^3} + \hat{\psi}\, y_1y_2y_3y_4y_5\ , \quad z = \hat{\psi}^{-3}\ .
\end{aligned} \tag{A.8}$$

The closed-string periods on the non-compact threefolds are solutions of the Picard-Fuchs operators

$$\mathcal{L}^{[n]} = \mathcal{L}_E^{[n]}(-z)\cdot\theta\,, \tag{A.9}$$

where $\mathcal{L}_E^{[n]}(z)$ denote the Picard-Fuchs operators for the representations of the elliptic curve E

$$\begin{aligned}
\mathbb{P}_{3,2,1}[6]: &\qquad \mathcal{L}_E^{[6]}(z) = \theta^2 - 12z(6\theta+5)(6\theta+1)\,,\\
\mathbb{P}_{2,1,1}[4]: &\qquad \mathcal{L}_E^{[4]}(z) = \theta^2 - 4z(4\theta+3)(4\theta+1)\,,\\
\mathbb{P}_{1,1,1}[3]: &\qquad \mathcal{L}_E^{[3]}(z) = \theta^2 - 3z(3\theta+1)(3\theta+2)\,,
\end{aligned} \tag{A.10}$$

with $\theta = z\frac{d}{dz}$. The equation for the elliptic curve is given by the restriction to $(y_4y_5)^n = -1$

k	$n_k^{[6]}$	$n_k^{[4]}$	$n_k^{[3]}$
1	16	8	2
3	-432	-24	-2
5	45 440	320	10
7	-7 212 912	-6 776	-84
9	1 393 829 856	175 536	858
11	-302 514 737 008	-5 123 448	-9 878
13	70 891 369 116 256	161 777 200	123 110
15	-17 542 233 743 427 360	-5 401 143 120	-1 622 890
17	4 520 954 871 206 554 016	187 981 969 232	22 308 658
19	-1 202 427 473 254 100 406 128	-6 756 734 860 408	-316 775 410
21	327 947 495 234 600 477 004 048	249 179 670 525 576	4 616 037 426
23	-91 298 034 448 725 882 319 078 384	-9 384 048 140 182 200	-68 700 458 258

Table 10: Disc invariants for the on-shell superpotentials $W_{\rm inst} = \frac{1}{2}T_{\rm inst}$ for the non-compact hypersurfaces $X^\flat$ of degree $d = 6, 4, 3$.

in (A.8).[25] Eq. (A.9) implies the relation $2\pi i\, \theta\Pi_\ell(z) = \pi_\ell(-z)$ between the periods $\Pi_\ell(z)$ of the non-compact threefold and the periods $\pi(z)$ on the elliptic curve.

A similar relation

$$2\pi i\, \theta T(z) = \tau(-z)\,, \tag{A.11}$$

holds for the chain integrals between the domain wall tension T of the non-compact threefold and the line integral τ of the associated elliptic curve E. They fulfill the inhomogeneous differential equation

$$\mathcal{L}^{[n]}T(z) \;=\; -\frac{c^{[n]}}{16\pi^2}\sqrt{z}\ , \qquad \mathcal{L}^{[n]}_E(z)\tau(z) \;=\; -\frac{c^{[n]}}{8\pi}\sqrt{z}\ , \tag{A.12}$$

in terms of the constants $c^{[n]}$

$$c^{[6]} = 16\ , \qquad c^{[4]} = 8\ , \qquad c^{[3]} = 2\ , \tag{A.13}$$

which determine the normalization of the of the domain wall tension T. Then the domain wall tensions T, which are now solutions to the normalized inhomogeneous Picard-Fuchs equations (A.12), contains the quantum instanton contribution $T_{\rm inst}$, which starts as

$$T_{\rm inst}(z) \;=\; -\frac{1}{2\pi^2}\left(c^{[n]}\sqrt{z}+\ldots\right)\ ,$$

and yields for the three geometries (A.8) the normalized disc invariants in Tab. 10.[26]

The normalization constants $c^{[n]}$ are determined by requiring integrality of the monodromy matrices with respect to the singularities of the moduli space of the extended period vector. The extended period vector consists of the bulk periods Π and the domain wall tension T. Alternatively, the constants $c^{[n]}$ can be determined by directly evaluating the line integral τ on the curve E and by exploiting its relation to the 3-chain integral T according to eq. (A.11). In the following we exemplify the two approaches for the non-compact sextic threefold (A.8) to determine the normalization constant $c^{[6]}$. The other two normalization constants $c^{[4]}$ and $c^{[3]}$ are obtained analogously.

[25] Keeping the convention eq. (2.22), the algebraic modulus of the Calabi-Yau manifold and the curve differ by a minus sign, as indicated in eq. (A.9) and below.

[26] Here we list the integral disc instanton numbers $n_k^{[n]}$. These invariants are related to the real invariants $n_{k,\rm real}^{[n]}$ in ref. [32] by a factor 2, *i.e.* $n_k^{[n]} = 2\cdot n_{k,\rm real}^{[n]}$.

The moduli space of the non-compact sextic threefold (A.8) exhibits three singularities $z=0$, $z=-\frac{1}{432}$, and $z=\infty$, which correspond to a large radius, a conifold, and a orbifold point of the moduli space. In the vicinity of the large radius point $|z|<\frac{1}{432}$ a complete set of solutions to the Picard-Fuchs operator $\mathcal{L}^{[6]}$ is given by

$$\begin{aligned}
\tilde{\Pi}_0(z) =&1 \ , \\
\tilde{\Pi}_1(z) =&\log z + \sum_{k=1}^{+\infty} \frac{(6k)!}{k!\,(2k)!\,(3k)!}\cdot\frac{(-z)^k}{k} \ , \\
\tilde{\Pi}_2(z) =&\frac{1}{2}(\log z)^2 + \sum_{k=1}^{+\infty} \frac{(6k)!}{k!\,(2k)!\,(3k)!}\cdot\frac{(-z)^k}{k}\cdot\bigg(\log z - \frac{1}{k} + 6\Psi(6k+1) \\
&\qquad -\Psi(k+1) - 2\Psi(2k+1) - 3\Psi(3k+1)\bigg) \ ,
\end{aligned} \tag{A.14}$$

in terms of the Polygamma function Ψ. Together with the solution $\tilde{T}$ to the inhomogeneous Picard-Fuchs equation $\mathcal{L}^{[6]}\tilde{T}(z)\sim\sqrt{z}$

$$\tilde{T}(z) \ = \ \frac{\pi}{32}\sqrt{z}\,\sum_{k=0}^{+\infty}\frac{\Gamma(6k+4)}{\Gamma(3k+\frac{5}{2})\Gamma(2k+2)\Gamma(k+\frac{3}{2})(k+\frac{1}{2})}(-z)^k \ , \tag{A.15}$$

they form the extended period vector $\tilde{\Pi}=\left(\tilde{\Pi}_0,\tilde{\Pi}_1,\tilde{\Pi}_2,\tilde{T}\right)$. For this vector we determine the large radius monodromy matrix $\tilde{M}_{\rm LR}$. Furthermore, by analytically continuation with the help of Barnes integrals to the other singular points in the moduli space we also infer the conifold and orbifold monodromy matrices $\tilde{M}_{\rm con}$ and $\tilde{M}_{\rm orb}$. Next we perform a change of basis to the integral extended period vector $\Pi=(\Pi_0,\Pi_1,\Pi_2,T)$ by demanding integrality of all the monodromy matrices. For the bulk sector these steps can be found in detail in ref. [48]. In addition to integrality of the monodromy matrices we require that the domain wall tension T vanishes at $z=\infty$. The latter condition arises because the domain wall tension T interpolates between two supersymmetric vacua that coincide at the orbifold point. After these steps we finally arrive at the integral periods

$$\begin{aligned}
\Pi_0(z) &= \tilde{\Pi}_0(z) \ = \ 1 \ , \\
\Pi_1(z) &= \frac{1}{2\pi i}\tilde{\Pi}_1(z) \ = \ t(z) \ , \\
\Pi_2(z) &= \frac{1}{(2\pi i)^2}\tilde{\Pi}_2(z) - \frac{1}{4\pi i}\tilde{\Pi}_1(z) - \frac{5}{12}\tilde{\Pi}_0(z) \ = \ \frac{1}{2}t(z)^2 - \frac{1}{2}t(z) - \frac{5}{12} + \Pi_{\rm inst}(z) \ , \\
T(z) &= \frac{32}{(2\pi i)^2}\tilde{T}(z) - \frac{1}{4\pi i}\tilde{\Pi}_1(z) + \frac{1}{4}\tilde{\Pi}_0(z) \ = \ -\frac{1}{2}t(z) + \frac{1}{4} + T_{\rm inst}(z) \ .
\end{aligned} \tag{A.16}$$

Here we also exhibit the classical terms in terms of the flat coordinate t and the instanton contributions $\Pi_{\rm inst}$ and $T_{\rm inst}$. In particular the normalized domain wall tension yields the normalized instanton contribution

$$T_{\rm inst}(z) \ = \ -\frac{16}{2\pi^2}\left(\sqrt{z} - \frac{512}{9}z^{3/2} + \frac{229\,376}{25}z^{5/2} - \ldots\right) \ ,$$

and hence the normalization constant $c^{[6]} = 16$ in eq. (A.13). The integral monodromy matrices in the integral basis (A.16) are then given by

$$M_{\mathrm{LR}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} , \quad M_{\mathrm{con}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad M_{\mathrm{orb}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} ,$$

with $M_{\mathrm{con}} M_{\mathrm{orb}} = M_{\mathrm{LR}}$.

As an independent calculation to determine normalized domain wall tensions we now directly reduce the three-chain integrals of the domain wall tensions between the curves $C^{\flat}_{\varepsilon,\pm}$ of eq. (3.31) to line integrals on the elliptic curve E. In order to evaluate the chain integrals we first change to the inhomogeneous coordinate α, y, z_1, z_2, which are suitable to evaluate the chain integrals [48]

$$y_1 = y - \frac{1}{2}\hat{\psi}\alpha z_1^3 z_2 \ , \quad y_2 = -\alpha z_1^2 \ , \quad y_3 = -i z_1 \ , \quad y_4 = -i z_2 \ , \quad y_5 = 1 \ .$$

In terms of these coordinates the hypersurface equation (A.8) of the non-compact sextic Calabi-Yau threefold $X^{\flat}$ becomes

$$X^{\flat} : \quad y^2 = z_1^6 \left(1 + \alpha^3 + \frac{1}{4}(\hat{\psi}\alpha)^2 z_2^2 \right) + (z_2^6 - 1) \ ,$$

while the holomorphic three form reads

$$\Omega(\hat{\psi}) = \frac{6}{(2\pi i)^3} \cdot \frac{\hat{\psi}\, z_1^2 \, d\alpha \, dz_1 \, dz_2}{2\sqrt{z_1^6 \left(1 + \alpha^3 + \frac{1}{4}(\hat{\psi}\alpha)^2 z_2^2\right) + (z_2^6 - 1)}} \ .$$

We can think of this geometry as a complex surface given in terms of the coordinates z_1 and z_2 fibered over a $\mathbb{P}^1$ base parametrized by the affine coordinate α. Furthermore, in these coordinates the curves $C^{\flat}_{\varepsilon,\kappa}$ are given by[27]

$$C^{\flat}_{\varepsilon,\kappa} = \left\{ z_1 = i\, z_2 \, , \ \alpha z_2 = -i\kappa\sqrt{\varepsilon\, i\, \hat{\psi}} \, , \ \varepsilon = \frac{i}{2}\hat{\psi}\alpha z_1^4 + y \right\} \ , \quad \varepsilon = \pm i \quad \kappa = \pm i \, .$$

The goal is now to evaluate the domain wall tensions

$$T_{\ell}(\hat{\psi}) = \int_{\Gamma_{\ell}} \Omega(\hat{\psi}) \ ,$$

where we consider the two three chains Γ_1 and Γ_2 bounded by

$$\partial\Gamma_1 = C_{+i,+i} - C_{-i,-i} \ , \qquad \partial\Gamma_2 = C_{-i,+i} - C_{-i,-i} \ .$$

As we will see in the calculation the domain wall tensions for the remaining combinations of curves do not yield independent results. The steps to reduce the three dimensional integral

[27] For ease of notation we have chosen here the explicit root $\eta = i$ for $\eta^6 = -1$ in eq. (3.31).

to a line integral over the $\mathbb{P}^1$ base are worked out and explained in detail in ref. [48]. Therefore for completeness we merely sketch the necessary steps here.

Instead of calculating the domain wall tension, it is easier to derive the line integral τ of eq. (A.11). With $z = \hat{\psi}^{-6}$ we get

$$\begin{aligned}\tau_\ell(-z) = 2\pi i\,\theta\, T_\ell(\hat{\psi}(z)) \;&=\; -\frac{2\pi i}{6}\,\hat{\psi}\,\frac{dT_\ell(\hat{\psi})}{d\hat{\psi}} \\ &= -\frac{\hat{\psi}}{(2\pi i)^2}\int_{\Gamma_\ell} d\alpha\, dz_2\, dz_1\,\frac{d}{dz_2}\frac{1}{2\sqrt{z_1^6\left(1+\alpha^3+\frac{1}{4}(\hat{\psi}\alpha)^2 z_2^2\right)+(z_2^6-1)}}\;.\end{aligned}$$

The simplification occurs because for the integrand the derivative with respect to $\hat{\psi}$ is equivalent to the derivative with respect to z_2. Then the integral over z_2 becomes trivial.[28] We now evaluate the integral over the coordinate z_1 along a closed contour encircling the six branch points of the square root. Next we integrate the coordinate z_2 along the interval from $z_2 = 1$ to $z_2 = -1$ to arrive at [48]

$$\int \Omega = -\frac{1}{2\pi i}\left(\int \frac{\hat{\psi}\, z_2\, d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}(\hat{\psi}z_2)^2\alpha^2}}\Bigg|_{z_2=1} - \int \frac{\hat{\psi}\, z_2\, d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}(\hat{\psi}z_2)^2\alpha^2}}\Bigg|_{z_2=-1}\right)\;. \tag{A.17}$$

Note that the performed integration is equivalent to the integration over a homology 2-sphere, as the contour in the z_1 coordinate can be shrunk to a point at the endpoints $z_2 = \pm 1$ of the interval.

If we now carry out the remaining integral (A.17) over α along a closed contour encircling the two branch points with leading behavior $\sim \hat{\psi}^{-1}$ for large $\hat{\psi}$, we integrate over a one cycle of the elliptic curve E and obtain the fundamental period of the elliptic curve

$$\pi_0(-z) \;=\; {}_2F_1(\tfrac{1}{6},\tfrac{5}{6};1;-432z) \;=\; 1 - 60z + 13\,860z^2 - \ldots\;.$$

If, instead, we reduce the three chains Γ_1 and Γ_2 to line integrals over α in eq. (A.17), we need to evaluate the integrals

$$\begin{aligned}\tau_1(-z) \;&=\; -\frac{1}{2\pi i}\left(\int_{i\sqrt{\hat{\psi}}}^{-\sqrt{\hat{\psi}}}\frac{\hat{\psi}\,d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}\hat{\psi}^2\alpha^2}} + \int_{-i\sqrt{\hat{\psi}}}^{\sqrt{\hat{\psi}}}\frac{\hat{\psi}\,d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}\hat{\psi}^2\alpha^2}}\right) \\ \tau_2(-z) \;&=\; -\frac{1}{2\pi i}\left(\int_{\sqrt{\hat{\psi}}}^{-\sqrt{\hat{\psi}}}\frac{\hat{\psi}\,d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}\hat{\psi}^2\alpha^2}} + \int_{-\sqrt{\hat{\psi}}}^{\sqrt{\hat{\psi}}}\frac{\hat{\psi}\,d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}\hat{\psi}^2\alpha^2}}\right)\end{aligned} \tag{A.18}$$

Here the integration boundaries for α are determined by requiring that the coordinates $(z_2 = \pm 1, \alpha)$ associated to the endpoints of the line integral correspond to a point on the appropriate curve $C^\flat_{\varepsilon,\kappa}$.

[28] Similarly as for the examples discussed in ref. [33], there is no contribution from the derivative $\frac{d}{d\hat{\psi}}$ acting on the three chain Γ_ℓ.

While the line integral (A.18) trivially vanishes for Γ_2, namely $\tau_2(-z) = 0$, we evaluate the integral over Γ_1 and arrive at

$$\tau(-z) \equiv \tau_1(-z) = \frac{16\sqrt{z}}{2\pi i}\, {}_3F_2(\tfrac{2}{3}, \tfrac{4}{3}, 1; \tfrac{3}{2}, \tfrac{3}{2}; -432z) - \frac{1}{2}\pi_0(-z) \ . \tag{A.19}$$

The resulting domain wall tension $\tau(-z) = 2\pi i\, \theta T(z)$ is in agreement with the result in eq. (A.16) and in eq. (A.12) together with the normalization $c^{[6]} = 16$ of eq. (A.13).

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